

Understanding games and linear programming through tropical geometry

PGMO internship proposal by Georg Loho and Mateusz Skomra

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Duration: Between 4 and 6 months

Place: LAAS-CNRS, Toulouse, France

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1 Overview

Oriented matroids are a powerful framework for studying questions from optimization and real algebraic geometry [3]. *Parity games* or more generally *mean payoff games* form well-studied classes of games with applications in machine verification and with an intriguing complexity status at the intersection of NP and co-NP [6]. The run of the simplex method for linear programming as well as the run of policy iteration for mean payoff games can be described by sign patterns forming an oriented matroid. The project will focus on contrasting the sign patterns arising for parity games, mean payoff games and for linear programming.

A geometric framework for analyzing these games and oriented matroids is provided by *signed tropical convexity*. In the project, we will use insights from tropical geometry in order to improve our understanding of mean payoff games. This gives rise to a second direction of research: extending the fundamentals of signed tropical convexity and its interpretation in the mentioned applications based on recent work [8].

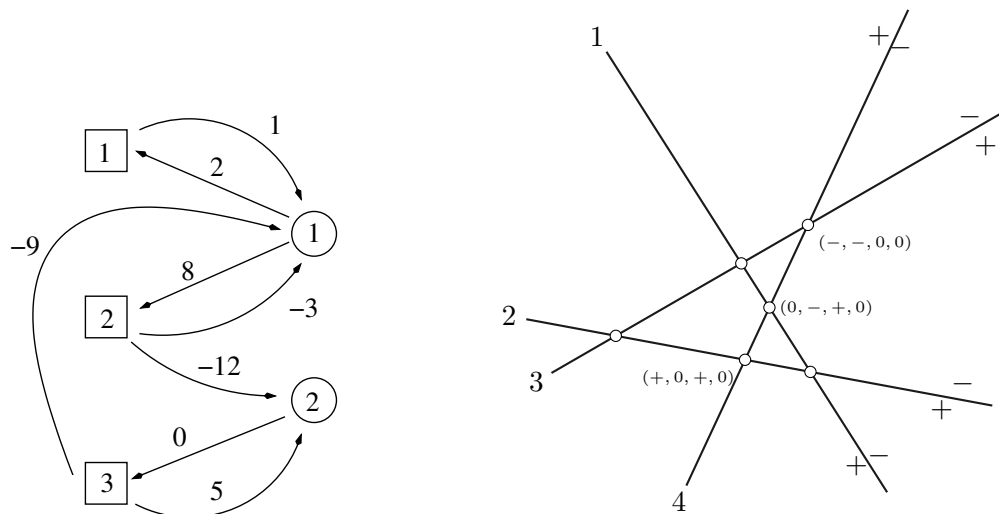


Figure 1: A game graph for a mean payoff game from [1] and lines corresponding to a 2-dimensional linear program with associated sign patterns from [7].

2 Construction

Already in early work, it was shown how one can reduce solving a linear program to the iteration over sign patterns [4]. More precisely, one can formulate the optimality conditions and the choice of locally improving directions purely in terms of the signs of the subdeterminants of an augmented constraint matrix. Axiomatizing the structure of these sign patterns lead to the introduction of oriented matroids.

In a similar way, one can associate sign patterns to a tropical linear program using the tropical analog of determinants [2, 5]. In the associated mean payoff game, this corresponds to signs of weights of directed cycles of the game graph. This allows to extract an oriented matroid directly from the mean payoff game which encodes the structure of the game. It is known that such an oriented matroid can also arise as the oriented matroid associated with a linear program.

3 Objectives

Depending on the interests of the student, the project may include the following subjects:

- Tropical vs. classical realizability of oriented matroids
- Policy iteration algorithms for parity games and their tropical interpretation
- Combinatorics and algorithms for signed tropical polyhedra

References

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- [4] Robert G. Bland. A combinatorial abstraction of linear programming. *J. Combinatorial Theory Ser. B*, 23(1):33–57, 1977.
- [5] Marcel Celaya, Georg Loho, and Chi Ho Yuen. Oriented matroids from triangulations of products of simplices, 2020. arXiv:2005.01787.
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- [7] Georg Loho. Abstract tropical linear programming. *Electron. J. Comb.*, 27(2):research paper p2.51, 33, 2020.
- [8] Georg Loho and Mateusz Skomra. Signed tropical halfspaces and convexity, 2022. arXiv:2206.13919.