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Oriented matroids and signed tropical convexity

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HIM

Overview

1. Basics of (signed) tropical calculations and inequality systems

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2. Oriented matroids and the (tropical) simplex method

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2. Oriented matroids and the (tropical) simplex method
3. Versions of signed tropical convexity and their relation with oriented matroids

Some success stories of tropical geometry

- Combinatorial approach to geometric questions: **Disproval of Ragsdale conjecture** (Itenberg-Viro 1996)
  Tool: Patchworking algebraic curves

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  Tool: combinatorial Hodge theory
- Limit-consideration for complexity questions: Certain log-barrier **interior point methods not strongly polynomial** (Allamigeon-Benchimol-Gaubert-Joswig 2018)
  Tool: tropical polytopes as limit of classical polytopes

Tropical semiring

Definition

Tropical numbers	$\mathbb{T}_{\geq \mathbb{0}} = \mathbb{R} \cup \{-\infty\}$
Addition	$s \oplus t := \max(s, t)$
Multiplication	$s \odot t := s + t$
Additive neutral	$\mathbb{0} = -\infty$

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Slogan: Replace $\sum$ by $\max$ and $\cdot$ by $+$

- $x \oplus y$ corresponds to $O(t^x) + O(t^y)$
- $x \odot y$ corresponds to $O(t^x) \cdot O(t^y)$

Tropical convexity

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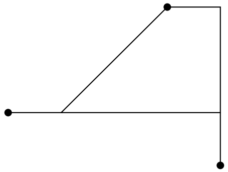
Example

$$0 \odot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus (-1) \odot \begin{pmatrix} 3 \\ 2 \end{pmatrix} \oplus (-1) \odot \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



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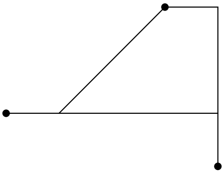

- No cancellation!!
- Only in tropical non-negative orthant...

(Non-negative) Tropical linear programming

Let $(a_{ji}), (b_{ji}) \in (\mathbb{R} \cup \{-\infty\})^{n \times d}$.

Theorem (too many references)

Checking if a system of the form

$$\max_{i \in [d]}(a_{ji} + x_i) \leq \max_{i \in [d]}(b_{ji} + x_i) \qquad \text{for } j \in [n]$$

has a solution $x \in \mathbb{R}^d$ is in NP $\cap$ co-NP.

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- Feasibility of these systems is equivalent with Mean Payoff Games (GKK 1988, MSS 2004, AGG 2012)
- Important subclass: Parity Games! – Quasipolynomial algorithms based on universal trees (Calude et al. 2017, Fijalkow, Jurdzinski, Czerwinski, Daviaud, Fijalkow, Jurdzinski, Lazic, Parys,. . .)

Signed tropical numbers

Symmetrized tropical semiring (ACGNQ 1990)

Signed tropical numbers	$\mathbb{T}_{\pm} = \mathbb{R} \cup \{\mathbb{O}\} \cup \ominus\mathbb{R}$ with $\mathbb{O} = -\infty$
Symmetrized tropical numbers	$\mathbb{S} = \mathbb{R} \cup \{\mathbb{O}\} \cup \ominus\mathbb{R} \cup \bullet\mathbb{R}$
$\oplus$	extension of $\max$
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Example

- $4 \oplus 4 = 4$
- $4 \oplus \ominus 4 = \bullet 4$
- $4 \oplus \ominus 5 = \ominus 5$
- $\ominus 4 \oplus \bullet 4 = \bullet 4$
- $3 \odot (\ominus 14) = \ominus 17$
- $\bullet - 11 \odot 99 = \bullet 88$
- $(\ominus 7 \oplus \ominus 16) \odot (\ominus - 19) = -3$

Trying to order the symmetrized tropical semiring

Bad news

- No compatible total order for the symmetrized tropical semiring
- No suitable equations

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Definition

- **Balance relation:** $x \bowtie y \quad \Leftrightarrow \quad x \ominus y \in \mathbb{T}_\bullet$
- **Strict partial order:** $x > y \quad \Leftrightarrow x \ominus y \in \mathbb{T}_{>\mathbb{0}}$
- **Pseudo-order:**
  $x \vDash y \quad \Leftrightarrow \quad x > y \text{ or } x \bowtie y \quad \Leftrightarrow \quad x \ominus y \in \mathbb{T}_{\geq \mathbb{0}} \cup \mathbb{T}_\bullet.$

- $1 \bowtie \bullet 6, \quad \bullet 6 \bowtie 3, \quad$ but $1 \not\bowtie 3$
- $-42 \vDash \ominus 100$
- $\bullet 3 \vDash \bullet 5$

Halfspaces

Let $(a_0, a_1, \ldots, a_d) \in \mathbb{T}_{\pm}^{d+1}$.

Definition (open signed (affine) tropical halfspace)

$$\mathcal{H}^{+}(a) = \left\{ x \in \mathbb{T}_{\pm}^{d} \;\middle|\; a \odot \binom{0}{x} > \mathbb{0} \right\}$$

Definition (closed signed (affine) tropical halfspace)

$$\overline{\mathcal{H}}^{+}(a) = \left\{ x \in \mathbb{T}_{\pm}^{d} \;\middle|\; a \odot \binom{0}{x} \vDash \mathbb{0} \right\}$$

Observation

The closed signed tropical halfspace $\overline{\mathcal{H}}^{+}(a)$ is the topological closure of the open signed halfspace $\mathcal{H}^{+}(a)$.

General tropical linear inequality systems

Theorem (Reformulation of $(\max, +)$ system)

The feasibility problem for systems of the form $A \odot x \dashv b$, $x \in \mathbb{T}_{\pm}^{d}$, $x \vDash \mathbb{0}$ is in NP $\cap$ co-NP.

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Theorem (L,Vegh 2020)

The feasibility problem for systems of the form $A \odot x \mathrel{\dashv} b$, $x \in \mathbb{T}_{\pm}^{d}$ is NP-complete.

General tropical linear inequality systems

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The feasibility problem for systems of the form $A \odot x \dashv b$, $x \in \mathbb{T}_{\pm}^{d}$ is NP-complete.

Proof.

Encode a formula

$$x_1 \vee \neg x_2 \vee \neg x_3$$

by

$$x_1 \oplus (\ominus x_2) \oplus (\ominus x_3) \vDash 0 \quad .$$

`True` corresponds to 0, `False` corresponds to $\ominus 0$.
Intersection of halfspaces gives $\wedge$ of clauses. □

Room for questions

Oriented matroids I

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -4 & -3 & -12 \\ 3 & -1 & 6 \\ 1 & 3 & 18 \\ -2 & 1 & 2 \end{pmatrix}$$

Chirotope:

$\mathrm{sign}\det(A_{123}) = -$

$\mathrm{sign}\det(A_{124}) = +$

$\mathrm{sign}\det(A_{125}) = -$

...

Vectors:

$\mathrm{sign}(1, -4, 3, 1, -2) = (+, -, +, +, -)$

$\mathrm{sign}(A_{\cdot,1} + A_{\cdot,3}) = (+, -, +, +, 0)$

...

Covectors:

(signs of vectors in $\ker(A^\intercal)$)

...

Oriented matroids II

Given $A \in \mathbb{R}^{d \times n}$ (with $d < n$), we have:
$\forall |X| = d - 1, Y = \{y_1, \ldots, y_{d+1}\}$,

$$\sum_{k=1}^{d+1} (-1)^k \det(A|_{X,y_k}) \det(A|_{Y \setminus y_k}) = 0.$$

$\Rightarrow$ Either + (positive) and - (negative) term, or they are all zeros.

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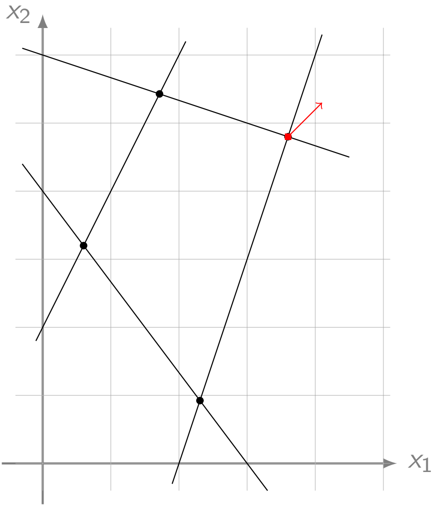
Chirotope

(Non-zero) map χ: $E^d \to \{+, -, 0\}$ with

- alternating, so essentially $\chi\binom{E}{d} \to \{+, -, 0\}$;
- Grassmann-Plücker: terms $(-1)^k \chi(X, y_k) \chi(Y \setminus y_k)$ either contain + and - term, or all zeros.

Geometric view on simplex method



$$\max \quad (1,1) \cdot x$$

$$\text{s. t.} \quad \begin{pmatrix} -4 & -3 \\ 3 & -1 \\ 1 & 3 \\ -2 & 1 \end{pmatrix} \cdot x \leq \begin{pmatrix} -12 \\ 6 \\ 18 \\ 2 \end{pmatrix}$$

$$x \geq 0$$

Simplex method with sign oracle

$A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n, c \in \mathbb{R}^d$

$$\begin{aligned} \max \quad & c^\intercal \cdot x \\ \text{s. t.} \quad & A \cdot x \leq b \\ & x \geq 0 \end{aligned}$$

Assumptions: Feasible region bounded, generic (simple) polytope

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For $J \in \binom{[n]}{d}$ set

$$x_J := A_J^{-1} b_J \qquad y_J := (A_J^\intercal)^{-1} c$$

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Require: $I \subseteq [n]$ with feasible x_I
  while y_I has negative entry **do**
    choose $i \in I$ with $y_i < 0$
    let $j \in [n]$ for which $J = I \setminus \{i\} \cup \{j\}$ defines feasible vertex x_J
    $I \leftarrow J$
  end while

Explicit calculations for running the simplex method

while y_I has negative entry **do**

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What is actually needed to run the algorithm?

y_I

$$A_I^\mathsf{T} y = c \overset{\text{Cramer's rule}}{\Leftrightarrow} y_j = \frac{\det(A_I^{(i)})}{\det(A_I)} \quad \left[A_I^{(i)} \text{ replacing } i\text{th row of } A_I \text{ with } c\right]$$

x_I

$$A \cdot A_I^{-1} b_I \leq b, A_I^{-1} b_I \geq 0$$

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end while

What is actually needed to run the algorithm?

Signs of subdeterminants of $\begin{pmatrix} A|b \\ I_d|0 \\ c^{\mathsf{T}}|0 \end{pmatrix}$

Oriented matroid from generic tropical matrix

Matrix $S \in \mathbb{T}_{\pm}^{n \times d}$ is *generic* iff, for each square submatrix M, the maximal absolute value is attained exactly once in

$$\mathrm{tdet}(M) = \bigoplus_{\sigma \in \mathrm{Sym}} \mathrm{tsgn}(\sigma) \prod_{i \in [n]} M_{i\sigma(i)} \ .$$

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Theorem (ABGJ 2015, Schewe 2009)

For each generic tropical matrix, there is a real matrix with the same chirotope.

$S \mapsto \left(\mathrm{tsgn}\,\mathrm{tdet}(S_J)\right)_{J\in\binom{[n]}{d}} \qquad A \mapsto \left(\mathrm{sign}\,\det(A_J)\right)_{J\in\binom{[n]}{d}}$

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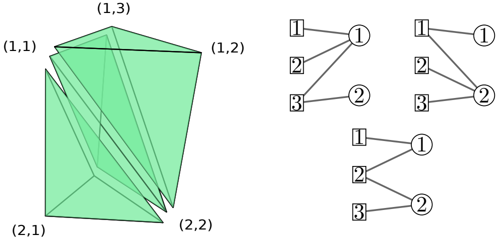
Theorem (Allamigeon, Benchimol, Gaubert, Joswig 2015)

For many pivot rules, the run of the simplex method on a tropical inequality system follows the path of a run on a linear inequality system. The tropical operations can be implemented efficiently.

Oriented matroid from triangulation I

Observation

The vertices of any full-dim simplex in $\triangle_{n-1} \times \triangle_{d-1}$ form a spanning tree.



- Axioms for simplices in the triangulation – give rise to *generic tropical oriented matroids*
- Matchings in the triangulation – abstraction of tropical determinant

Oriented matroid from triangulation II

Generic tropical matrix gives rise to regular triangulation by using them as weights on the vertices of $\triangle_{n-1} \times \triangle_{d-1}$

Theorem (L 2017)

One can run the tropical simplex method purely with the data of the triangulation.

Theorem (Celaya,L,Yuen 2020+)

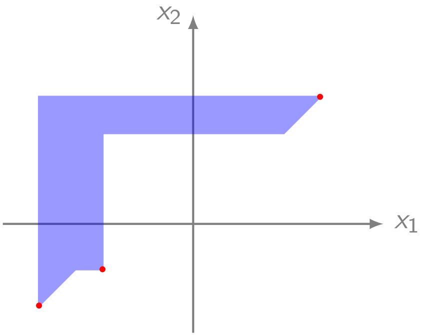
The matchings together with a sign matrix give rise to an oriented matroid.

Room for questions

Versions of signed convexity – TO-convexity

$$\operatorname{conv}_{\mathrm{TO}}(A) := \bigcup \Big\{ \mathcal{U}(A \odot x) \colon x \in \mathbb{T}^n_{\geq \mathbb{0}}, \bigoplus_{j \in [n]} x_j = 0 \Big\}$$

with $\mathcal{U}(b) := \begin{cases} [\ominus|b|, |b|] & \text{for } b \in \mathbb{T}_\bullet \\ b & \text{else} \end{cases}$.



$\operatorname{conv}_{\mathrm{TO}}\left(\{(3,3), (\ominus 1, \ominus 0), (\ominus 4, \ominus 2)\}\right)$

$$(-3) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} \ominus 1 \\ \ominus 0 \end{pmatrix} = \begin{pmatrix} \ominus 1 \\ \bullet 0 \end{pmatrix}$$

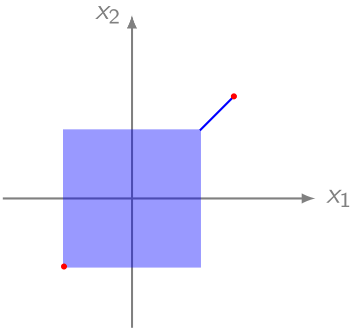
$$(-2) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} \ominus 1 \\ \ominus 0 \end{pmatrix} = \begin{pmatrix} \bullet 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} \oplus (-1) \odot \begin{pmatrix} \ominus 4 \\ \ominus 2 \end{pmatrix} = \begin{pmatrix} \bullet 3 \\ 3 \end{pmatrix}$$

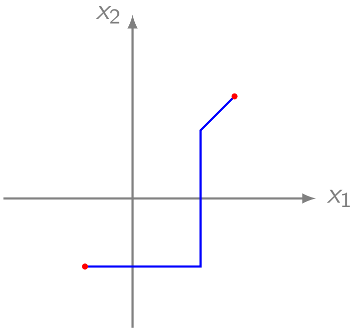
$$(-1) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} \ominus 4 \\ \ominus 2 \end{pmatrix} = \begin{pmatrix} \ominus 4 \\ \bullet 2 \end{pmatrix}$$

Basic properties of TO-convexity

- Intersection preserves convexity
- Coordinate projection preserves convexity
- Hull operator
- Tropically convex if and only if line segments are contained

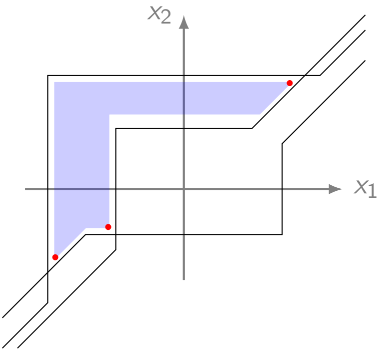


$\mathrm{conv}_{\mathrm{TO}}((0,0),(\ominus-2,\ominus-2))$



$\mathrm{conv}_{\mathrm{TO}}((0,0),(\ominus-3,\ominus-2))$

Representation by halfspaces



Theorem (L,Végh 2020)

The TO-convex hull of finitely many points coincides with the intersection of all **open** *tropical halfspaces that contain them.*

Versions of signed convexity – TC-convexity

Left sum

$$\lhd\colon \mathbb{T}_{\pm}^2 \to \mathbb{T}_{\pm}\,, \quad a \lhd b = \begin{cases} a & \text{if } |a| \geq |b|, \\ b & \text{if } |b| > |a|. \end{cases}$$

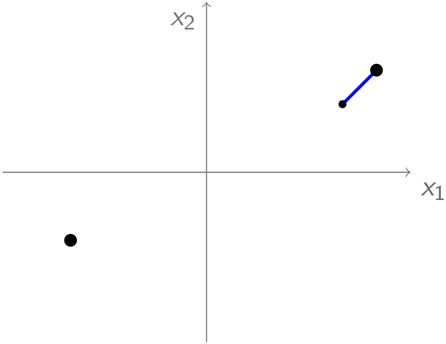
Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{T}_{\pm}^d$ be an n-element set.

$\mathrm{Faces}(x_1, \ldots, x_n) :=$ union of faces of $\mathcal{U}(x_1, \ldots, x_n)$ with vertices in $\{x_{\sigma(1)} \lhd \ldots \lhd x_{\sigma(n)} \colon \sigma \in \mathrm{Sym}(n)\}$.

TC-convex hull

$$\mathrm{conv}_{\mathrm{TC}}(X) := \bigcup \Big\{ \mathrm{Faces}(\lambda_1 \odot x_1, \ldots, \lambda_n \odot x_n) \colon \lambda \in \mathbb{T}_{\geq 0}^n, \bigoplus_i \lambda_i = 0 \Big\}$$

Representation by halfspaces – TC-convexity



Theorem (L,Skomra 2021+)

The TC-convex hull of finitely many points coincides with the intersection of all **closed** *tropical halfspaces that contain them.*

Oriented matroids via (co)vectors

Sign vectors $\mathcal{L} \subseteq \{\ominus 0, \mathbb{0}, 0\}^E$ called *covectors* satisfying:

(CV0) $\mathbb{0} \in \mathcal{L}$

(CV1) $X \in \mathcal{L} \iff \ominus X \in \mathcal{L}$

(CV2) $X, Y \in \mathcal{L} \implies X \triangleleft Y \in \mathcal{L}$

(CV3) For $X, Y \in \mathcal{L}$, $e \in S(X, Y)$, $\exists\ Z \in \mathcal{L}$ with

- $Z_e = \mathbb{0}$
- For all $f \notin S(X, Y)$, we have $Z_f = (X \triangleleft Y)_f$, where $S(X, Y) = \{f \in E \colon X_f = \ominus Y_f \neq \mathbb{0}\}$.

Oriented matroids via (co)vectors

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(CV0) $\mathbb{O} \in \mathcal{L}$

(CV1) $X \in \mathcal{L} \iff \ominus X \in \mathcal{L}$

(CV2) $X, Y \in \mathcal{L} \implies X \triangleleft Y \in \mathcal{L}$

(CV3) For $X, Y \in \mathcal{L}$, $e \in S(X, Y)$, $\exists\ Z \in \mathcal{L}$ with

- $Z_e = \mathbb{O}$
- For all $f \notin S(X, Y)$, we have $Z_f = (X \triangleleft Y)_f$, where $S(X, Y) = \{f \in E \colon X_f = \ominus Y_f \neq \mathbb{O}\}$.

Theorem (Red Book)

An oriented matroid is equivalently given by

- *a chirotope,*
- *a set of vectors,*
- *a set of covectors.*

Real Bergman fan

$$\Sigma_M = \bigcup \mathrm{conv}\{\mathbf{e}_{X_1}, \ldots, \mathbf{e}_{X_k}\}$$

Union over comformal flag $X_1 \leq \ldots \leq X_k$ of signed covector

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Union over comformal flag $X_1 \leq \ldots \leq X_k$ of signed covector

Signed matroid polytope

$$P_M^{\pm} = \mathrm{conv}\{e_X \colon X \text{ signed basis of } M\} \subseteq \mathbb{R}^E$$

Real Bergman fan

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Signed matroid polytope

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Observation

An oriented matroid is equivalently given by

- its real Bergman fan,
- its signed matroid polytope.

Real Bergman fan

$$\Sigma_M = \bigcup \mathrm{conv}\{\mathbf{e}_{X_1}, \ldots, \mathbf{e}_{X_k}\}$$

Union over comformal flag $X_1 \leq \ldots \leq X_k$ of signed covector

Signed matroid polytope

$$P_M^{\pm} = \mathrm{conv}\{e_X \colon X \text{ signed basis of } M\} \subseteq \mathbb{R}^E$$

Observation

An oriented matroid is equivalently given by

- its real Bergman fan,
- its signed matroid polytope.

Theorem (Celaya 2019)

The real Bergman fan Σ_M is a well-behaved subfan of the normal fan of the signed matroid polytope $P_M^{\pm}$.

Real Bergman fan as TC-convex hull

signed cocircuits $\mathcal{C}$ – support-minimal covectors of M
signed circuits $\mathcal{D}$ – support-minimal vectors M

Note: The support of a signed (co)circuit is a (co)circuit.

Observation

For circuit $c \in \mathcal{C}$ and cocircuit $d \in \mathcal{D}$ holds $c \odot d \in \mathbb{T}_{\bullet}$.

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Theorem (Celaya 2020, L, Skomra 2021+)

$$\mathrm{slog}(\Sigma_M) \cap [\ominus 0, 0]^E = \mathrm{conv}_{\mathrm{TC}}(\mathcal{C}) = \bigcap_{d \in \mathcal{D}} \overline{\mathcal{H}}^{+}(\mathbb{O}, d) \cap [\ominus 0, 0]^E$$

Matroids over Hyperfields

A *hyperfield* $(\mathbb{H}, \boxplus, \otimes, 0, 1)$ is a field-like algebraic structure, but $\boxplus : \mathbb{H} \times \mathbb{H} \to 2^{\mathbb{H}}$ can be multi-valued.

- $a \boxplus b \boxplus c := \bigcup_{x \in a \boxplus b} x \boxplus c = \bigcup_{y \in b \boxplus c} a \boxplus y$.
- For $a \in \mathbb{H}$, $-a$ is the unique element such that $0 \in a \boxplus (-a)$.

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Definition (Baker, Bowler 2017)

A **strong** matroid over $\mathbb{H}$ is an alternating $\chi : L^d \to \mathbb{H}$ such that

$$0 \in \boxplus_{k=1}^{d+1} (-1)^k \chi(X, y_k) \otimes \chi(Y \setminus y_k).$$

A **weak** matroid only requires the 3-term GP as long as $\underline{\chi}$ is a matroid.

Examples

- A field $K \Rightarrow$ (Represented) K-subspaces
- Krasner hyperfield $\mathbb{K} = \{0, 1\}$, $1 \boxplus 1 = \{0, 1\}$.
  $\Rightarrow$ Usual matroids.
- Sign hyperfield: $\mathbb{S} = \{+, -, 0\}$, $+ \boxplus - = \{+, -, 0\}$.
  $\Rightarrow$ Oriented matroids.
- Tropical hyperfield: $\mathbb{T} = \mathbb{R} \cup \{\infty \approx {}''0''\}$, $a \boxplus b = \min\{a, b\}$ if $a \neq b$, $a \boxplus a = [a, \infty]$, and $a \otimes b = a + b$.
  $\Rightarrow$ Valuated matroids/tropical linear spaces.
- There exist hyperfields s.t. $\{$strong matroids$\} \subsetneq \{$weak matroids$\}$.

Connection with hyperoperations

Definition (*real plus-tropical hyperfield* $\mathbb{TR}$ (Viro 2010))

- additive hyperoperation on $\mathbb{T}_\pm$ given by

$$x \boxplus y = \begin{cases} \operatorname{argmax}_{x,y}(|x|,|y|) & \text{if } \chi \subseteq \{+,\mathbb{0}\} \text{ or } \chi = \{-\} \\ [\ominus|x|,|x|] & \text{else} \end{cases} .$$

- multiplicative group $(\mathbb{T}_\pm, \odot)$

Example

- $2 \boxplus \ominus 3 = \ominus 3$
- $3 \boxplus \ominus 3 = [\ominus 3, 3]$

- TO-convexity and hyperfields
- TC-convexity and hyperfields

Room for questions

Research questions I

Oriented matroids arising from parity games (PG), mean payoff games (MPG), linear programming and triangulations (see above):

Research questions I

Oriented matroids arising from parity games (PG), mean payoff games (MPG), linear programming and triangulations (see above):

- Are there structural differences for these classes?
- Are the ones from PG & MPG particularly difficult or particularly easy?

Research questions II

Signed tropical linear inequality systems:

- When can we decide feasibility in polynomial time (includes MPG)?
- Can we identify when they are not (NP-)hard?

Research questions III

Signed convexity and LP algorithms:

- Tropicalization of different forms of LP
- Oriented matroids and interior point methods (geometry of central path and circuit imbalance measure)

Further references

- Thesis of Stéphane Gaubert
- Thesis of Pascal Benchimol
- Thesis of Oliver Friedmann
- Oriented Matroids from Triangulations of Products of Simplices - Marcel Celaya, L, Chi Ho Yuen
- Signed tropical convexity - L, László A. Végh
- Work in progress with Mateusz Skomra
- Thesis of Marcel Celaya