Neural Networks for Regular Matroid Polytopes

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Further reading

Arithmetic Circuits and Neural Networks for Regular Matroids
arXiv:2511.02406

Motivation

- judge computational power of neural networks: "can we solve everything with neural networks"
- subtraction free computation as interesting model: we want to compute the basis generating polynomials of matroids
- understanding classes of matroids (in particular regular and binary matroids)

New insights

- refined decomposition of regular matroids: "nodes of regular matroids generalizing nodes of graphic matroids"
- NNs for regular matroid optimization (vs binary matroids): representing the support function of matroid base polytope
- matroid polytopes: xc vs vxc

Prerequisites

Polytopes

Given as convex hull of vertices or (bounded) solution set of linear inequalities

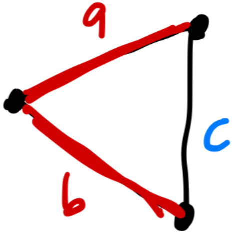
Matroids

Set of bases fulfilling some axioms

Examples of matroids

- Sets of columns of a matrix forming basis of the column span
- Spanning trees of a connected graph

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$



Computational models

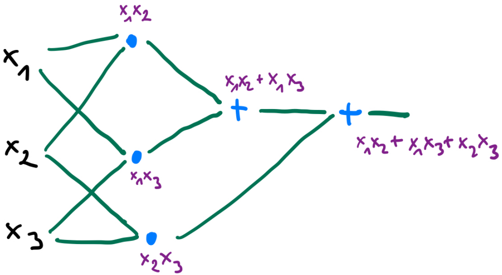
arithmetic circuit: directed acyclic graph defining an arithmetic expression using 2-ary operations like max, min, $+$, $-$, $\times$, or $/$.

- representing polynomials: $(\cdot, +, \cdot \mathit{const})$
- tropical circuits: $(\max, +)$; $(\max, +, +\mathit{const})$
- subtraction-free circuits: $(\cdot, +, /)$
- tropicalized subtraction-free circuit $(\max, +, -)$

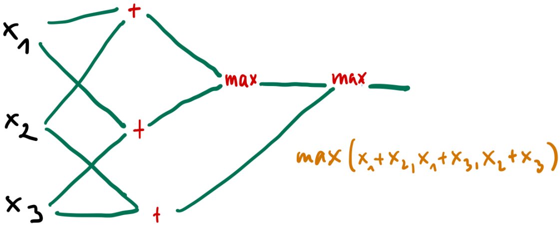
tropicalization replace

- $+$ by max
- $\cdot$ by $+$
- $/$ by $-$

Classical arithmetic circuit



Tropical circuit



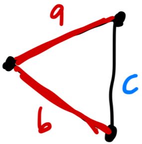
Basis generating polynomial

Let $\mathcal{B}$ be the set of bases of a matroid.

$$\sum_{B \in \mathcal{B}} x^B = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e$$

Example:

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$



$$x_a x_b + x_a x_c + x_b x_c$$

Note: this is a Lorentzian polynomial.

Support function of matroid base polytope

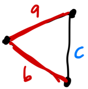
Let $\mathcal{B}$ be the set of bases of a matroid.
Tropicalization of

$$\sum_{B \in \mathcal{B}} x^B$$

is

$$\max_{B \in \mathcal{B}} \chi(B)^\top z$$

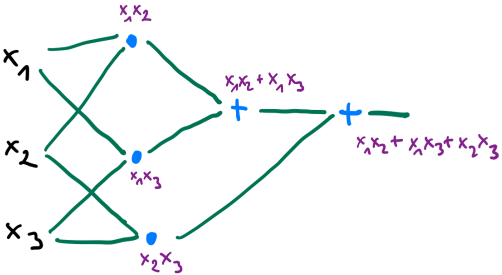
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 0 & 1 & 1 \end{array}$$



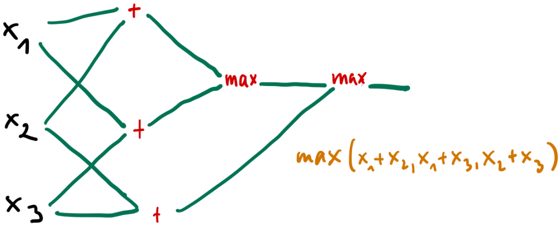
$$\max(x_a + x_b, x_a + x_c, x_b + x_c)$$

Note: this is the support function of the matroid base polytope.

Classical arithmetic circuit



Tropical circuit

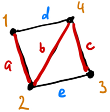


Regular matroids

Equivalent characterizations:

- realizability over all fields (2 and 3 enough)
- Tutte: all determinants of realization in $\{-1, 0, 1\}$
- binary (no $U(2,4)$-minor) and no Fano or dual Fano $\{F_7, F_7^\star\}$
- Seymour decomposition: 1-,2-,3-sums of graphic matroids or cographic matroid or R_{10}

Examples of regular matroids



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

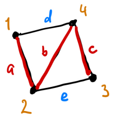
make full rank and normalize:

$$\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

binary realization

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Examples of regular matroids



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binary realization

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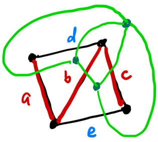
looking at the dual



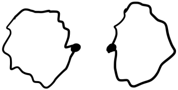
$$\begin{array}{c} \begin{array}{ccccc} d & e & a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Neither graphic nor co-graphic

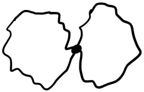
$(I|A^{10})$ and $(I|A^{12})$ for

$$A^{10} := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad A^{12} := \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \; .$$

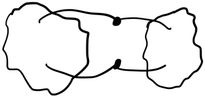
1-sum



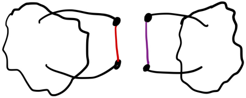
yields



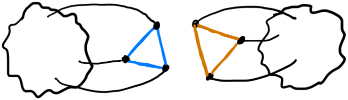
2-sum



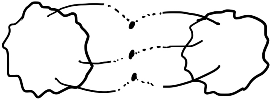
yields



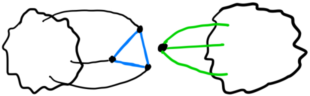
3-sum



yields



Glueing with co-graphic:



Technical details about the sums

Generalization of ‘clique-sum’

Let

- M_1 and M_2 binary matroids,
- E the disjoint union of the ground sets $E = E(M_1) \triangle E(M_2)$.

Then there is a matroid $M_1 \triangle M_2$ with ground set E whose set of circuits consists of the minimal non-empty subsets of E of the form $X_1 \triangle X_2$ where X_i is a disjoint union of circuits of M_i.

(Generalized) Matrix Tree Theorem

Given

- regular matroid M with
- full-rank totally unimodular realization $A \in \{0, \pm 1\}^{r \times n}$.

Let $X := \mathrm{diag}(x_e : e \in E(M))$ and $L := AXA^\intercal$.

Theorem (Maurer 1976)

The basis generating polynomial of M is $\det(L)$.

A basic algorithm for basis generating polynomial of regular matroid

Input: independence oracle

- Construct realization over $\mathbb{F}_2$: construct basis, determine fundamental circuits
- Use Camion's "signing" algorithm to find realization A over $\mathbb{R}$.
- Compute the determinant $\det(AXA^\intercal)$.

A basic algorithm for basis generating polynomial of regular matroid

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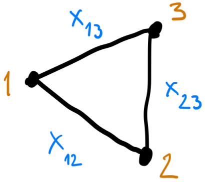
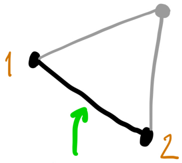
Problem: we want *subtraction-free* computation

Idea for graphic matroids



Set $y_3 = \max(x_{13}, x_{23})$.



$y_3 + \max(x_{12}, x_{13} + x_{23} - y_3)$

Note that

$$\max(x_{12} + x_{13}, x_{12} + x_{23}, x_{13} + x_{23}) =$$
$$\max(x_{13}, x_{23}) + \max(x_{12}, x_{13} + x_{23} - \max(x_{13}, x_{23}))\,.$$

Generalized star-mesh transformation

Let

- $A \in \{0, \pm 1\}^{r \times n}$,
- $N_1 := \mathrm{supp}(A_r)$ (support of rth row of A)
- $N_0 := [n] \setminus N_1$

Definition

Star-mesh transformation on A with respect to r: $A' \in \mathbb{R}^{(r-1) \times n'}$, where $n' := |N_0| + \binom{|N_1|}{2}$ with

- $A'_{i,j} := A_{i,j}$ if $i \in [r-1]$, $j \in N_0$,
- $A'_{i,(j,k)} := A_{i,j} - (A_{r,j} A_{r,k} A_{i,k})$ for $i \in [r-1]$ and $j, k \in N_1$ with $j < k$.

Theorem (Hertrich, Kober, L. 2025+)

The generalized star-mesh transformation yields a procedure to compute the basis generating polynomial with a smaller matrix.

Refined Seymour decomposition

Seymour decomposition

Every regular matroid M can be decomposed into

- graphic matroids,
- cographic matroids,
- matroids isomorphic to R_{10}

by repeated 1-, 2-, and 3-sum decompositions.

Key local-global insight (Bérczi, Mátravölgyi, Schwarcz 2024)

Let M be a 3-connected regular matroid, such that M is not graphic, cographic or isomorphic to R_{10}.
Then there are 3-connected regular matroids M_1 and M_2, such that $|E(M_2)| \geq 9$, M_2 is graphic and $M_1 \oplus_3 M_2 \in \{M, M^\star\}$.

Inductive conclusion

Theorem (HKL 2025+)

For a regular matroid M with n elements, there is a $(+, \times, /)$-circuit of size $O(n^3)$ computing the basis generating polynomial of M.

Corollary (HKL 2025+)

For a regular matroid M with basis set $\mathcal{B}$ and n elements, there is

- a $(\max, +, -)$-circuit and
- a ReLU neural network of size $O(n^3)$

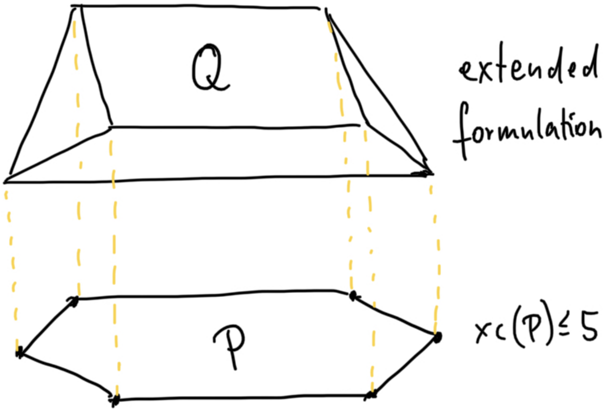
computing the tropical polynomial $\max_{B \in \mathcal{B}} \sum_{e \in B} x_e$.

Limitation

It is hard to compute the number of bases of binary matroids (the same as evaluating basis generating polynomial at 1 or also special evaluation of Tutte polynomial)

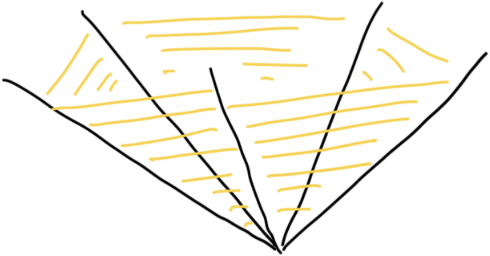
Extension complexity

- Let $Q \subset \mathbb{R}^e$ and $P \subset \mathbb{R}^d$ with $e \geq d$ and $\phi(Q) = P$ for an affine projection π. Then Q is an *extended formulation* of P.
- *extension complexity*
  $\mathrm{xc}(P) = \min\{\text{number of facets of } Q \mid \pi(Q) = P\}$

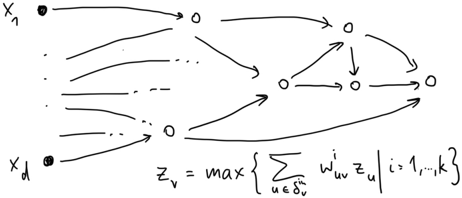


(Monotone) NNs are extended formulations

epigraph $\mathrm{epi}(f) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x)\}$



$$y_v \geq \sum_{u \in \delta_v^{\mathrm{in}}} w_{uv}^i y_u \quad \forall\, i = 1, \dots, k \quad \forall \text{ maxout units } v \;\; .$$



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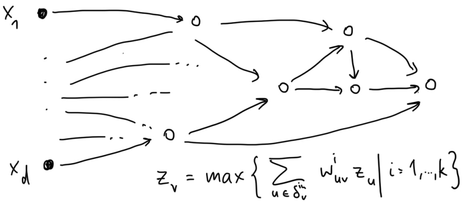

$$y_v \geq \sum_{u \in \delta_v^{\text{in}}} w_{uv}^i y_u \quad \forall\ i = 1, \dots, k \quad \forall \text{ maxout units } v \ .$$

Crucial observation:

$$y_v \geq \sum_{u \in \delta_v^{\text{in}}} w_{uv}^i y_u \quad \forall\ i = 1, \dots, k \quad \Leftrightarrow \quad y_v \geq \max_{i=1,\dots,k} \sum_{u \in \delta_v^{\text{in}}} w_{uv}^i y_u \ .$$

Proposition (Hertrich, L. 2024+)

If $f \colon \mathbb{R}^d \to \mathbb{R}$ is represented by a *monotone* rank-k maxout network with $s \geq 1$ maxout units, then $\operatorname{xc}(\operatorname{epi}(f)) \leq ks$.

Virtual extension complexity

$$\mathrm{vxc}(P) = \min\{\mathrm{xc}(Q) + \mathrm{xc}(R) \mid P + Q = R\}$$

Observation

- extension complexity can go down by Minkowski addition
- there are matroid polytopes with exponential extension complexity but they are all summands of the regular permutahedron is $O(n \log(n))$

Theorem (Hertrich, Kober, L. 2025+)

The virtual extension complexity of regular matroids is $O(n^3)$.

This is in contrast to the currently best known extension complexity upper bound of $O(n^6)$ (Aprile, Fiorini 2022).

Conclusion

Understanding of neural networks using combinatorics and polyhedral geometry!

- tropicalization
- structure theory of regular matroids
- neural networks as computational model
- connection to extension complexity