Signed tropical convexity

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Motivation

- Tropical linear programming equivalent to mean payoff games; feasibility in NP $\cap$ co-NP but no polynomial-time algorithm known (Akian, Gaubert, Guterman 2012)
- Intimate connection between classical linear programming and tropical linear programming (Schewe 2009, Allamigeon, Benchimol, Gaubert, Joswig 2015+)
- Many statements for classical polytopes have natural formulation when containing the origin

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Further connections

- Quest for a strongly polynomial algorithm for linear programming (Smale 1998)
- Modeling scheduling problems through tropical linear programming (Butkovic 2010)
- Bijection between regular subdivisions of products of simplices and tropical point configurations (Develin, Sturmfels 2004)

Overview on Polytopes

- Polytopes as convex hull of finitely many points
- Duality between containment in a convex hull and linear programming
- Farkas' Lemma for convex hull

Tropical inequality systems

Let $(a_{ji}), (b_{ji}) \in (\mathbb{R} \cup \{-\infty\})^{n \times d}$.

Theorem (GKK 1988, MSS 2004, AGG 2012)

Checking if a system of the form

$$\max_{i \in [d]}(a_{ji} + x_i) \leq \max_{i \in [d]}(b_{ji} + x_i) \qquad \textit{for } j \in [n]$$

has a solution $x \in \mathbb{R}^d$ is in NP $\cap$ co-NP.

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has a solution $x \in \mathbb{R}^d$, where we are allowed to swap a_{ji} with b_{ji} for some $i \in [d]$, is NP-complete.

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Theorem (L,Vegh 2019+)

The feasibility problem for systems of the form $A \odot x \dashv b$, $x \in \mathbb{T}_{\pm}^d$ is NP-complete.

Tropical semiring

Definition

Tropical numbers	$\mathbb{T}_{\geq \mathbb{0}} = \mathbb{R} \cup \{-\infty\}$
Addition	$s \oplus t := \max(s, t)$
Multiplication	$s \odot t := s + t$
Additive neutral	$\mathbb{0} = -\infty$

Operations are extended componentwise to $\mathbb{T}^d$

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Example

$$(5 \oplus -7) \odot 10 \oplus -100 = 15$$

$$(-3) \odot x \oplus 1 = 9 \quad \text{valid for } x = 12$$

$$\text{But: } (-3) \odot x \oplus 9 = 9 \quad \text{valid for every} \quad x \leq 12$$

Example

$$0 \odot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus (-1) \odot \begin{pmatrix} 3 \\ 2 \end{pmatrix} \oplus (-1) \odot \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

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Definition (ACGNQ 1990)

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Symmetrized tropical numbers	$\mathbb{S} = \mathbb{R} \cup \{\mathbb{0}\} \cup \ominus\mathbb{R} \cup \bullet\mathbb{R}$
Non-negative	$\mathbb{T}_{\geq \mathbb{0}} = \mathbb{R} \cup \{\mathbb{0}\} = \{x \in \mathbb{S} \colon x \geq \mathbb{0}\}$
Non-positive	$\mathbb{T}_{\leq \mathbb{0}} = \ominus\mathbb{R} \cup \{\mathbb{0}\} = \{x \in \mathbb{S} \colon x \leq \mathbb{0}\}$
Balanced	$\mathbb{T}_{\bullet} = \bullet\mathbb{R}$

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Balanced	$\mathbb{T}_{\bullet} = \bullet\mathbb{R}$
Addition	$x \oplus y = \begin{cases} \mathrm{argmax}_{x,y}(\lvert x\rvert, \lvert y\rvert) & \text{if } \lvert\chi\rvert = 1 \\ \bullet\, \mathrm{argmax}_{x,y}(\lvert x\rvert, \lvert y\rvert) & \text{else} \end{cases}.$
Multiplication	$x \odot y = (\mathrm{tsgn}(x) * \mathrm{tsgn}(y))\,(\lvert x\rvert + \lvert y\rvert)$

where

- $\lvert\,.\,\rvert \colon \mathbb{S} \to \mathbb{R} \cup \{\mathbb{0}\}$ removes the sign,
- $\mathrm{tsgn}(\,.\,) \colon \mathbb{S} \to \{\oplus, \ominus, \bullet, \mathbb{0}\}$ recalls only the sign,
- $\chi = \{\mathrm{tsgn}(\xi) \mid \xi \in (\mathrm{argmax}(\lvert x\rvert, \lvert y\rvert))\}$.

Calculating with signed tropical numbers

One can think of computation with complexity classes in the sense

- $x \oplus y$ corresponds to $O(t^x) + O(t^y)$
- $x \odot y$ corresponds to $O(t^x) \cdot O(t^y)$

Example

- $4 \oplus 4 = 4$
- $4 \oplus \ominus 4 = \bullet 4$
- $\ominus 4 \oplus \bullet 4 = \bullet 4$
- $3 \odot (\ominus 14) = \ominus 17$
- $\bullet - 11 \odot 99 = \bullet 88$
- $(\ominus 7 \oplus \ominus 16) \odot (\ominus - 19) = -3$

Trying to order the symmetrized tropical semiring

Bad news

- No compatible total order for the symmetrized tropical semiring
- No suitable equations

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Definition

- **Balance relation:** $x \bowtie y \quad \Leftrightarrow \quad x \ominus y \in \mathbb{T}_\bullet$
- **Strict partial order:** $x > y \quad \Leftrightarrow x \ominus y \in \mathbb{T}_{>\mathbb{0}}$
- **Pseudo-order:** $x \vDash y \quad \Leftrightarrow \quad x > y$ or $x \bowtie y \quad \Leftrightarrow \quad x \ominus y \in \mathbb{T}_{\geq \mathbb{0}} \cup \mathbb{T}_\bullet$.

Example

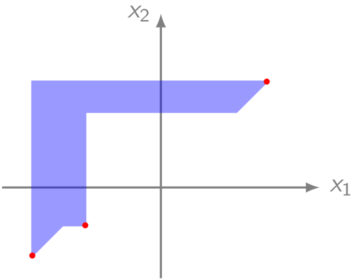
- $1 \bowtie \bullet 6, \quad \bullet 6 \bowtie 3, \quad$ but $1 \not\bowtie 3$
- $-42 \vDash \ominus 100$
- $\bullet 3 \vDash \bullet 5$

Signed tropical convex hull – I

Definition (Inner hull)

$$\operatorname{tconv}(A) = \left\{ z \in \mathbb{T}_{\pm}^{d} \;\middle|\; z \bowtie A \odot x, x \in \mathbb{T}_{\geq 0}^{n}, \bigoplus_{j\in[n]} x_j = 0 \right\} \subseteq \mathbb{T}_{\pm}^{d}$$

$$= \bigcup \left\{ \mathcal{U}(A \odot x) \;\middle|\; x \in \mathbb{T}_{\geq 0}^{n}, \bigoplus_{j\in[n]} x_j = 0 \right\} \text{ with } \mathcal{U}(a) := [\ominus|a|, |a|] \, .$$



$\operatorname{tconv}(\{(3,3), (\ominus 1, \ominus 0), (\ominus 4, \ominus 2)\})$

$$(-3) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} \ominus 1 \\ \ominus 0 \end{pmatrix} = \begin{pmatrix} \ominus 1 \\ \bullet 0 \end{pmatrix}$$

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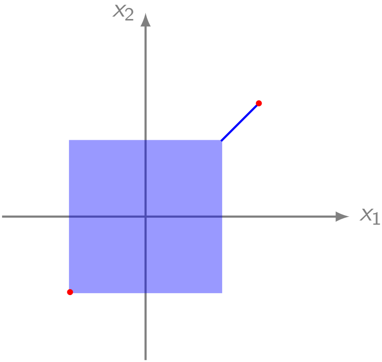
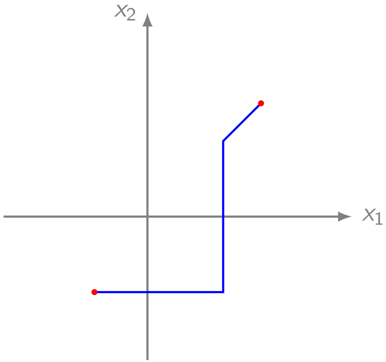
Signed tropical convex hull – II

Basic properties

- Intersection preserves convexity
- Coordinate projection preserves convexity
- Hull operator
- Tropically convex if and only if line segments are contained



$\mathrm{tconv}((0,0),(\ominus -2, \ominus -2))$



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Duality

Let $A = (a_{ij}) \in \mathbb{T}_{\pm}^{d \times n}$ and $b \in \mathbb{T}_{\pm}^{d}$.

Definition (Non-negative kernel)

$$\ker_{+}(A) = \left\{ x \in \mathbb{T}^{n}_{\geq \mathbb{0}} \setminus \{\mathbb{0}\} \;\middle|\; A \odot x \bowtie \mathbb{0} \right\}$$

The origin $\mathbb{0}$ is in the convex hull $\mathrm{tconv}(A)$ if and only if the non-negative kernel $\ker_{+}(A)$ is not empty.

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Definition (open tropical cone)

$$\operatorname{sep}_{+}(A) = \{ y \in \mathbb{T}_{\pm}^{d} \mid y^{\top} \odot A > \mathbb{0} \} \ .$$

It contains the separators of the columns of A from the origin.

Farkas' lemma

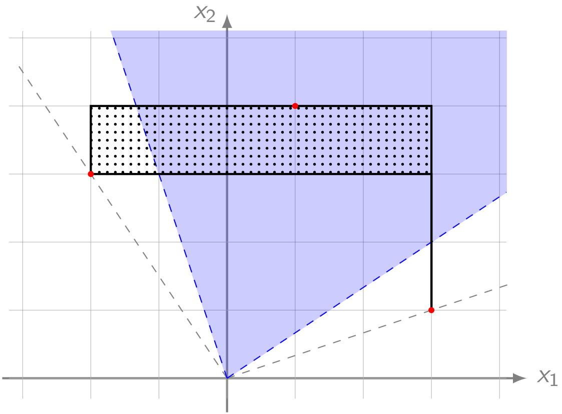
Theorem

For a matrix $A \in \mathbb{T}_{\pm}^{d\times n}$ exactly one of the sets $\ker_+(A)$ and $\operatorname{sep}_+(A)$ is nonempty.

Proof.

- New version of Fourier-Motzkin elimination
- Construction of explicit separator

□



exp-image of trop. conv. hull

Halfspaces

Let $(a_0, a_1, \ldots, a_d) \in \mathbb{T}_{\pm}^{d+1}$.

Definition (open signed (affine) tropical halfspace)

$$\mathcal{H}^+(a) = \left\{ x \in \mathbb{T}_{\pm}^d \;\middle|\; a \odot \begin{pmatrix} 0 \\ x \end{pmatrix} > \mathbb{0} \right\} .$$

Definition (closed signed (affine) tropical halfspace)

$$\overline{\mathcal{H}}^+(a) = \left\{ x \in \mathbb{T}_{\pm}^d \;\middle|\; a \odot \begin{pmatrix} 0 \\ x \end{pmatrix} \in \mathbb{T}_{\geq \mathbb{0}} \cup \mathbb{T}_{\bullet} \right\} . \tag{1}$$

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- Open tropical halfspaces are tropically convex.
- Closed tropical halfspaces are not tropically convex.

Observation

The closed signed tropical halfspace $\overline{\mathcal{H}}^{+}(a)$ is the topological closure of the open signed halfspace $\mathcal{H}^{+}(a)$.

Interlude - Encoding SAT

Theorem (L,Vegh 2019+)

The feasibility problem for systems of the form $A \odot x \dashv b$, $x \in \mathbb{T}_{\pm}^{d}$ is NP-complete.

Proof.

Encode a formula

$$x_1 \vee \neg x_2 \vee \neg x_3$$

by

$$x_1 \oplus (\ominus x_2) \oplus (\ominus x_3) \vDash 0 \ .$$

`True` corresponds to 0, `False` corresponds to $\ominus 0$.
Intersection of halfspaces gives $\wedge$ of clauses. □

Representation by Halfspaces - I

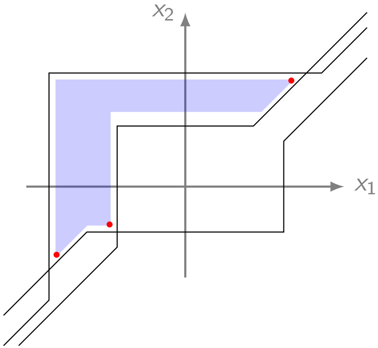
Theorem

For a matrix $A \in \mathbb{T}_{\pm}^{d \times n}$, the intersection of the open halfspaces containing their columns agrees with their tropically convex hull, that means

$$\operatorname{tconv}(A) = \bigcap_{A \subseteq \mathcal{H}^+(v)} \mathcal{H}^+(v) \quad \textit{for all suitable } (v_0, v_1, \ldots, v_d) \in \mathbb{T}_{\pm}^{d+1} .$$

Proof.

Careful use of Farkas' Lemma □



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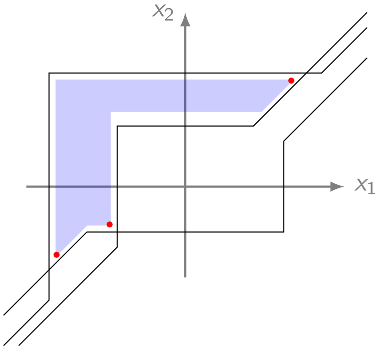
Separation works for strict inequalities!

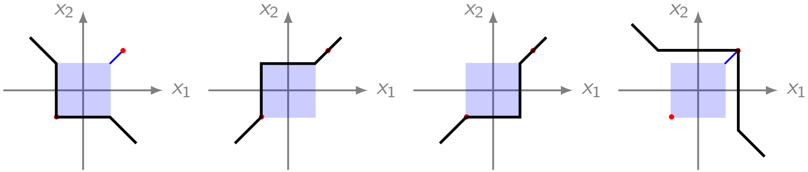


Representation by Halfspaces - II

Theorem (Minkowski-Weyl theorem)

For each finite set $V \subset \mathbb{T}_{\pm}^{d}$, there are finitely many closed tropical halfspaces H such that $\operatorname{tconv}(V)$ is the intersection of the halfspaces.

For each finite set H of closed halfspaces, whose intersection M is tropically convex, there is a finite set of points $V \in \mathbb{T}_{\pm}^{d}$ such that $M = \operatorname{tconv}(V)$.



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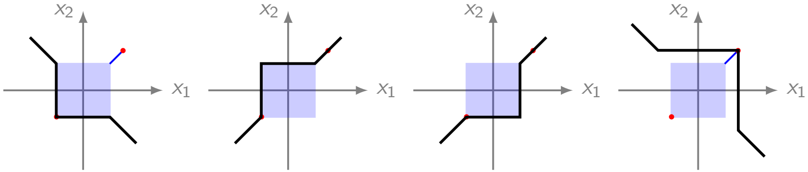
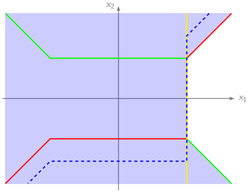


Proof.

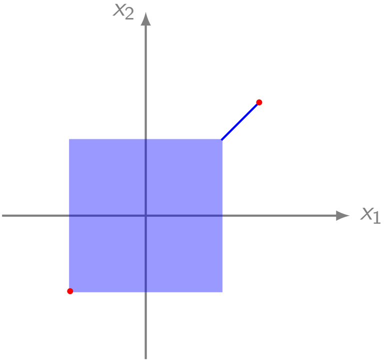
Suitably resolving balanced entries. □



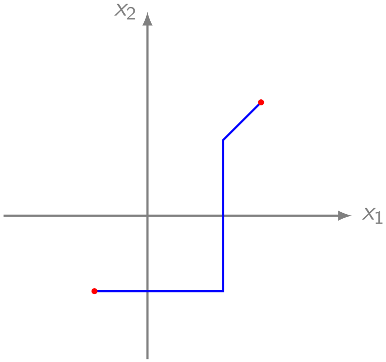
Convexity in each orthant

Theorem

A tropically convex set is the union of the tropically convex sets spanned by its intersection with the boundary of an orthant.



tconv((0, 0), (⊖ − 2, ⊖ − 2))



tconv((0, 0), (⊖ − 3, ⊖ − 2))

Connection to Puiseux polyhedra

Puiseux series $\mathbb{R}\{\{t\}\}$
valuation val (maps an element to its leading exponent)

Example

$\operatorname{val}(\pi t^4 - 100t^{-2.3}) = 4, \qquad \operatorname{val}(0) = -\infty$

Sign information: $\operatorname{sgn}\colon \mathbb{R}\{\{t\}\} \to \{\ominus, \mathbb{0}, \oplus\}$
Signed valuation: $\operatorname{sval}\colon \mathbb{R}\{\{t\}\} \to \mathbb{T}_{\pm}$
maps an element $k \in \mathbb{R}\{\{t\}\}$ to $\operatorname{sgn}(k)\operatorname{val}(k)$.

Lemma

One can define polytopes over $\mathbb{R}\{\{t\}\}$ like over $\mathbb{R}$.

Theorem

The signed hull $\operatorname{tconv}(A)$ is the union of the signed valuations for all possible lifts

$$\operatorname{tconv} A = \bigcup_{\operatorname{sval}(\mathbf{A})=A} \operatorname{sval}(\operatorname{conv}(\mathbf{A})) \ .$$

Connection with hyperoperations

Definition (*real plus-tropical hyperfield* $\mathbb{H}$ (Viro 2010))

- additive hyperoperation on $\mathbb{T}_{\pm}$ given by

$$x \boxplus y = \begin{cases} \operatorname{argmax}_{x,y}(|x|, |y|) & \text{if } \chi \subseteq \{+, \mathbb{0}\} \text{ or } \chi = \{-\} \\ [\ominus|x|, |x|] & \text{else} \end{cases} .$$

- multiplicative group $(\mathbb{T}_{\pm}, \odot)$

Example

- $2 \boxplus \ominus 3 = \ominus 3$
- $3 \boxplus \ominus 3 = [\ominus 3, 3]$

Theorem

$$\operatorname{tconv}(A) = A \boxdot \Delta_n := \left\{ A \boxdot x \;\middle|\; \bigoplus_{j \in [n]} x_j = 0, x \geq \mathbb{0} \right\} \subset \mathbb{T}_{\pm}^d .$$

Conclusion

Summary

- Extended notion of tropical convexity for signed tropical numbers
- New phenomena (strict vs. non-strict inequalities)
- Duality and elimination work (essentially)
- Representation by generators and halfspaces

Further Work

- Combinatorial study of signed tropical polytopes
- Linear programming without non-negativity constraints
- Feasibility check w.r.t. the origin