Tropical Volume by Tropical Ehrhart Polynomials

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Introduction

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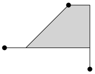
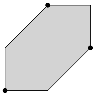
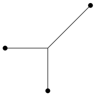

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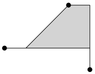
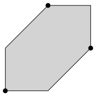
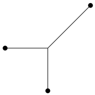

Recent studies show that metric tropical concepts are useful:

- log-barrier methods (Allamigeon, Benchimol, Gaubert & Joswig, 2018)
- tropical Voronoi diagrams (Criado, Joswig & Santos, 2019)
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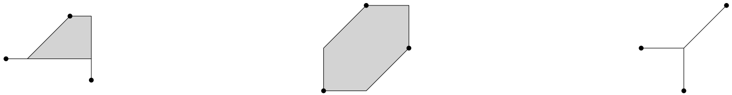

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Main goal: Identify an instrinsic volume concept for tropical polytopes.

Review of classical volume

Let $P \subseteq \mathbb{R}^d$ be a polytope. The classical volume concept for P is the Lebesgue measure:

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Second discretization:

Theorem (Ehrhart, 1967)

If P is an integral polytope, that is, all vertices are from $\mathbb{Z}^d$, then

$$\#\left(kP \cap \mathbb{Z}^d\right) = c_d(P)k^d + c_{d-1}(P)k^{d-1} + \ldots + c_1(P)k + c_0(P), \quad \text{for } k \in \mathbb{N}.$$

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Idea: Retrieve concept of *tropical volume* by turning this around – tropically.

Natural idea: Tropical integers could be $\mathbb{TN} := \mathbb{Z}_{\geq 0} \cup \{-\infty\}$.

Tropical lattices and tropical lattice polytopes

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☞ $\mathbb{TN}^d \subseteq \bigcap_{b \in \mathbb{N}_{\geq 2}} \log_b(\mathbb{Z}_{\geq 0})^d$

Tropical Ehrhart polynomial

Theorem (L & Schymura, 2019+)

Let $b \in \mathbb{N}_{\geq 2}$ *and let* $P \subseteq \mathbb{T}^d$ *be a tropical lattice polytope. Then, for* $k \in \mathbb{Z}_{\geq 0}$*, the tropical lattice point enumerator* $\mathfrak{L}_P^b(k) = \#\left((k \odot P) \cap \log_b(\mathbb{Z}_{\geq 0})^d\right)$ *agrees with a polynomial in* b^k *of degree at most* d*.*

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We write

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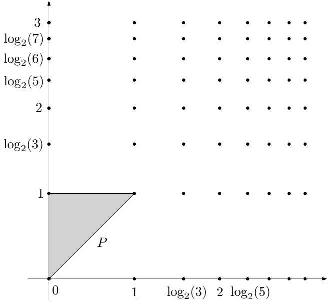

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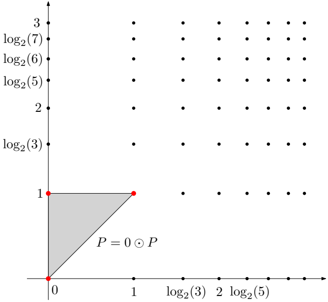

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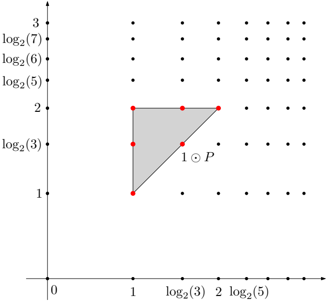

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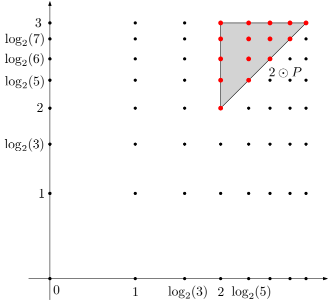

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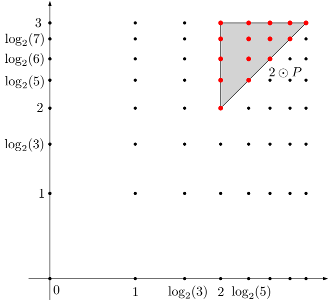

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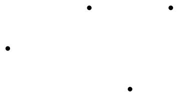
$$\mathfrak{L}_P^2(k) = \tfrac{1}{2} \cdot (2^k)^2 + \tfrac{3}{2} \cdot 2^k + 1$$

Interpretation of a tropical polytope P as a polytopal complex (Develin & Sturmfels).



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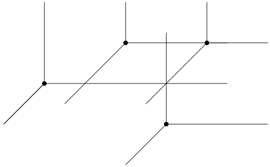

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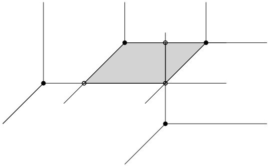

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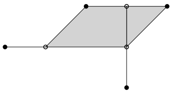

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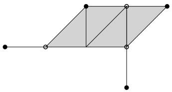

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If P is a tropical lattice polytope, then $\mathcal{C}_P$ can be refined into a triangulation $\mathcal{T}_P$ consisting of *alcoved simplices*, which are faces and lattice translates of

$$\Delta_\pi(\mathbf{0}) := \operatorname{conv}\left\{\mathbf{0}, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \ldots, e_{\pi(1)} + \ldots + e_{\pi(d)} = \mathbf{1}\right\},$$

where $\pi \in S_d$ is a permutation on $[d]$.

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Main lemma

Let $k \in \mathbb{Z}_{\geq 0}$. For $a \in \mathbb{Z}^d_{\geq 0}$ and $b \in \mathbb{N}_{\geq 2}$ write $D^a_b = \operatorname{diag}(b^{a_1}, \ldots, b^{a_d}) \in \mathbb{Z}^{d \times d}$. Then, the map $\phi : \mathbb{R}^d_{>0} \to \mathbb{R}^d$ defined by $\phi(z) = (\log_b(z_1), \ldots, \log_b(z_d))$ induces a bijection between

$$\left(b^k D^a_b \mathbf{1} + (b-1) b^k D^a_b \Delta(\mathbf{0})\right) \cap \mathbb{Z}^d_{\geq 0} \quad \text{and} \quad (k \odot \Delta(a)) \cap \log_b(\mathbb{Z}_{\geq 0})^d.$$

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- $D^a_b \Delta(\mathbf{0})$ is a classical lattice polytope $\Rightarrow k \mapsto \#\left((b-1) b^k D^a_b \Delta(\mathbf{0}) \cap \mathbb{Z}^d\right)$ is a polynomial in b^k

Tropical Ehrhart polynomial – how to?

Idea: Do ordinary Ehrhart theory on (transformed) alcoved simplices and stitch together.

By symmetry we look at $\Delta(\mathbf{0}) := \Delta_{id}(\mathbf{0}) = \operatorname{conv}\{\mathbf{0}, e_1, e_1 + e_2, \ldots, e_1 + \ldots + e_d\}$.

Main lemma

Let $k \in \mathbb{Z}_{\geq 0}$. For $a \in \mathbb{Z}^d_{\geq 0}$ and $b \in \mathbb{N}_{\geq 2}$ write $D^a_b = \operatorname{diag}(b^{a_1}, \ldots, b^{a_d}) \in \mathbb{Z}^{d \times d}$. Then, the map $\phi : \mathbb{R}^d_{>0} \to \mathbb{R}^d$ defined by $\phi(z) = (\log_b(z_1), \ldots, \log_b(z_d))$ induces a bijection between

$$\left(b^k D^a_b \mathbf{1} + (b-1) b^k D^a_b \Delta(\mathbf{0})\right) \cap \mathbb{Z}^d_{\geq 0} \quad \text{and} \quad (k \odot \Delta(a)) \cap \log_b(\mathbb{Z}_{\geq 0})^d.$$

- use $\Delta(\mathbf{0}) = \left\{x \in \mathbb{R}^d : 0 \leq x_d \leq \ldots \leq x_1 \leq 1\right\}$ and that $\log_b(\cdot)$ is strictly increasing
- $D^a_b \Delta(\mathbf{0})$ is a classical lattice polytope $\Rightarrow k \mapsto \#\left((b-1) b^k D^a_b \Delta(\mathbf{0}) \cap \mathbb{Z}^d\right)$ is a polynomial in b^k
- decomposition of P into alcoved simplices gives polynomiality of $k \mapsto \mathfrak{L}^b_P(k)$ and information on its coefficients $c^b_i(P)$

Cells of different dimensions

Definition (Trunk)

The *trunk* Trunk(P) of a tropical polytope P is defined as

$$\mathrm{Trunk}(P) := \bigcup \{F \in \mathcal{C}_P : \exists G \in \mathcal{C}_P \text{ with } \dim(G) \geq d \text{ such that } F \subseteq G\}\,.$$

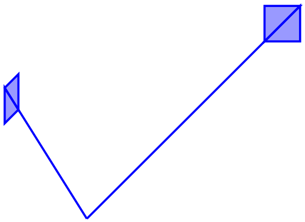

Figure: A 4-dimensional tropical polytope whose 2-trunk is disconnected.

Proposition (L & Schymura, 2019+)

The tropical convex hull of two full-dimensional pure tropical polytopes is a pure, full-dimensional tropical polytope.
Consequently, the d-trunk of a tropical polytope in $\mathbb{T}^d$ is a tropical polytope.

The logarithm map of a function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is

$$\operatorname{Log}|f| := \lim_{b \to \infty} \log_b |f(b)|.$$

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Tropical barycentric volume

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$$\operatorname{Log}|c_d^b(P)| = \max\{x_1 + \ldots + x_d : x \in \operatorname{Trunk}(P)\}.$$

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Definition (Tropical barycentric volume)

The *tropical barycentric volume* of a tropical polytope $P \subseteq \mathbb{T}^d$ is defined as

$$\operatorname{tbvol}(P) := \max\{x_1 + \ldots + x_d : x \in \operatorname{Trunk}(P)\}.$$

Corollary

The tropical barycentric volume is the sum of the coordinates of the barycenter of its d-trunk.

Let $\mathcal{P}_{\mathbb{T}}^d$ be the family of tropical polytopes in $\mathbb{T}^d$.
The function tbvol : $\mathcal{P}_{\mathbb{T}}^d \to \mathbb{T}$ has the following properties:

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5. *Non-singularity:* $\operatorname{tbvol}(P) = -\infty$ *if and only if* $\operatorname{Trunk}(P) = \emptyset$.

Comparison with existing tropical volume concepts

Depersin, Gaubert & Joswig: For $A \in \mathbb{T}^{d\times(d+1)}$, let $\operatorname{tvol}(A) = |\operatorname{tdet}(\bar{A}) - \operatorname{tdet}_\sigma(\bar{A})|$.

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Definition (Depersin, Gaubert & Joswig, 2017)

For a matrix $M \in \mathbb{T}^{d\times m}$ its *upper dequantized tropical volume* is defined as

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☞ If P is pure, that is, $P = \operatorname{Trunk}(P)$, then $\operatorname{tbvol}(P) = \operatorname{qtvol}^+(M)$.

A matrix $S \in \mathbb{T}^{r \times r}$ is *non-singular* if the value of the tropical determinant is attained at most once. The *tropical rank* $\mathrm{trk}(M)$ of a matrix $M \in \mathbb{T}^{d \times m}$ is the size of a largest non-singular square submatrix of M.

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Lemma (L & Schymura, 2019+)

Let $M \in \mathbb{TN}^{d \times m}$ and let $P = \operatorname{tconv}(M)$. Then,

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Question

How fast can we compute $\operatorname{tbvol}(P) = \operatorname{Log} |c_d^b(P)|$?

If P is pure, then $\operatorname{tbvol}(P) = \operatorname{qtvol}^+(M)$.
☞ $\operatorname{tbvol}(P)$ can be computed in time $O(n^3)$ (Depersin, Gaubert & Joswig, 2017).

Proposition

Computing the tropical barycentric volume $\operatorname{tbvol}(P)$ *is at least as hard as checking feasibility of a tropical linear inequality system (which is in NP ∩ coNP).*

Outlook

Future directions:

- (tropical Ehrhart positivity) Lower bounds on $\operatorname{Log}|c_i^b(P)|$ in terms of (non-negative) generalized tropical volumes tbvol_i.
- (special polytopes) Tropical Ehrhart polynomials of k^{th} tropical hypersimplex $\Delta_k^d = \operatorname{tconv}\left\{\sum_{j\in J} e_j : J \in \binom{[d]}{k}\right\}$.
- (discrete tropical surface area) Find geometric interpretation of $\operatorname{Log}|c_{d-1}^b(P)|$.
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