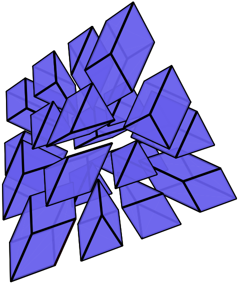
Combinatorics of Tropical Linear Programming

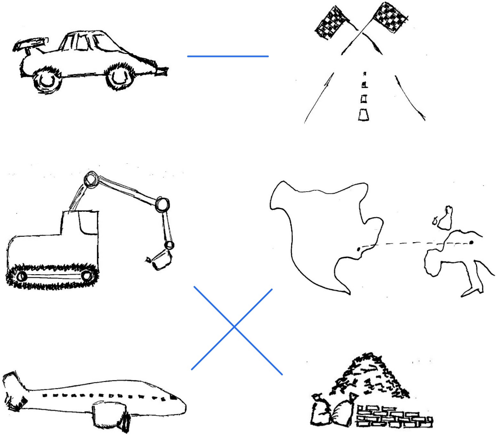
Georg Loho



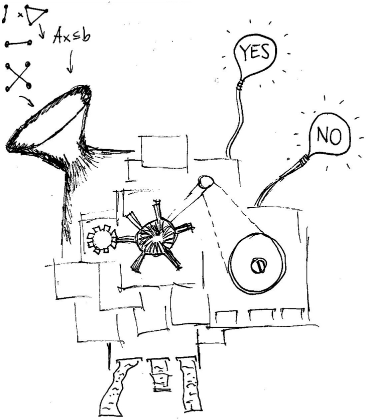
Berlin, September 5th 2017

Related Problems

(Combinatorial) Optimization



Complexity Questions



Motivation

- Connection with *Classical Linear Programming* / *Simplex Method* (Allamigeon, Benchimol, Gaubert, Joswig 2014+)
- *Oriented Matroid Programming* (Bland '77, Fukuda '82, Todd '85, Terlaky '85)

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- Equivalence of Tropical Linear Programming and *Mean Payoff Games* (Akian, Gaubert, Guterman 2012)
- Unclear complexity of Tropical Linear Programming in *NP* $\cap$ *co-NP* (Jurdziński '98)

Simplex Method (Dantzig '63, Bland '77,...)

Given $A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n$. Consider

$$A \cdot x \leq b \ . \tag{1}$$

Problem: Find a *feasible point* $y \in \mathbb{R}^d$ fulfilling system (1).

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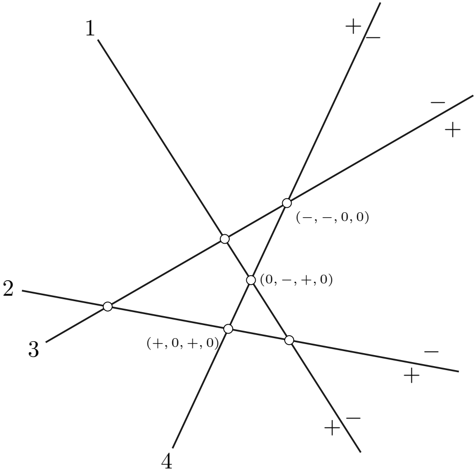
$I \leftarrow$ appropriate d-elem. subset of $[n]$ (*basis* of rows)
$y \leftarrow$ solution of $A_I \cdot x = b_I$
while y does not fulfill (1) and no certificate for infeasibility found **do**
    $f \leftarrow$ particular element of $[n] \setminus I$
    $e \leftarrow$ particular element of I
    $I \leftarrow I \cup \{f\} \setminus \{e\}$ (again a *basis*)
    $y \leftarrow$ solution of $A_I \cdot x = b_I$
end while
return y

Use dual objective vector, reduced cost vector

Halfspace Arrangement in $\mathbb{R}^2$



Classically: Abstraction by sign vectors, dual sign vectors (Oriented Matroid)

From Classical to Tropical Linear Programming

- Linear Programming over field of rational functions (Jeroslow '73)
- Tropical polyhedra are exactly the images of polyhedra over Puiseux series under valuation map (Develin, Yu 2007)
- Tropicalization of the Simplex Method (ABGJ 2014)
- Field of Puiseux Fractions suitable for computations (polymake) (Joswig, L, Lorenz, Schröter 2016)

Theorem (JLLS 2016)

The combinatorial type of a real polyhedron obtained by substituting a sufficiently large real number for the parameter equals the combinatorial type of the corresponding polyhedron of Puiseux fractions.

- Dequantization through logarithmic limit (Maslov '86)

Cramer Theorems

Fix $I \subset [n]$ of size d.

Classical (Cramer 1750)

$$(A_I | - b_I) \cdot x = 0 \ \ .$$

Solution vector given by $d \times d$-subdeterminants of $(A_I | - b_I)$.

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Tropical ('Max Plus' '97, AGG 2014)

$\min$ attained twice per row in $M_I \odot_{\min} x = (\min_{j \in [d+1]} (m_{ij} + x_j))_{i \in I}$

- Solution given by tropical $d \times d$-subdet. of $M \in (\mathbb{R} \cup \infty)^{n \times (d+1)}$.
- Tropical determinants are minimal $d \times d$ matchings.

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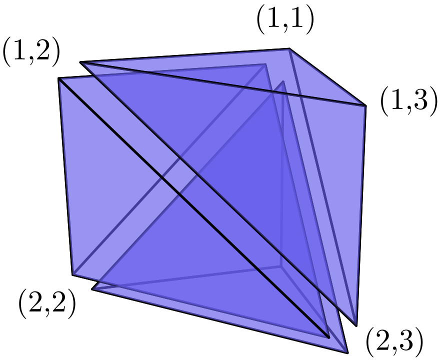
Abstract Tropical (L 2017)

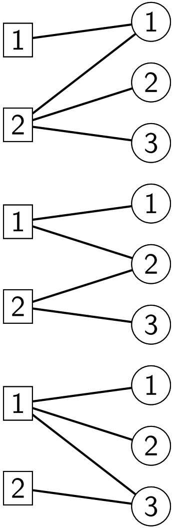
Cramer covector of I: Bipartite tree on $[d+1] \sqcup [n]$ such that d nodes in $I \subset [n]$ have degree 2 (and the others are of degree 1).

- Generalization of '$\min$ attained twice', special *covector graphs*.

Subpolytopes of $\Delta_d \times \Delta_{n-1}$ and Bipartite Graphs

$\Delta_1 \times \Delta_2$





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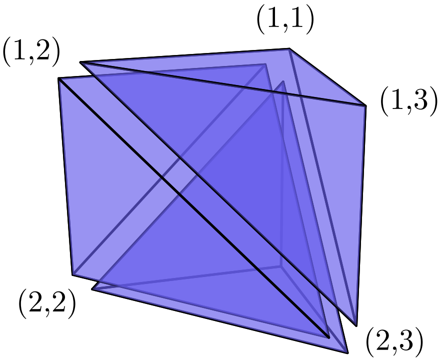

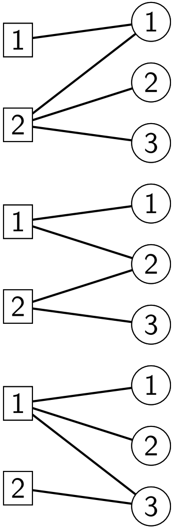

Theorem (DS 2004, FR 2015, JL 2016)

The covector graphs describe a regular subdivision of a subpolytope of $\Delta_d \times \Delta_{n-1}$.

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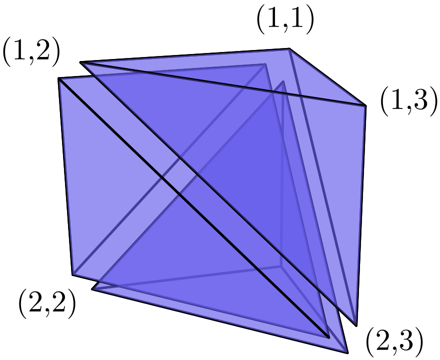

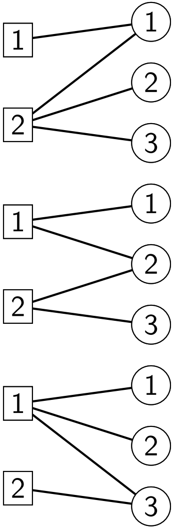

Definition (Signed tropical matroid (L 2017))

Bipartite graphs, which form a *not-necessarily regular subdivision* of a subpolytope of $\Delta_d \times \Delta_{n-1}$, with signs on the edges.

Abstract Tropical Cramer Theorem

Theorem (Postnikov 2009, L 2017)

For a given d-set $I \subseteq [n]$, there is exactly one full-dimensional cell in a triangulation of $\Delta_d \times \Delta_{n-1}$ so that in the corresponding bipartite graph the d nodes in I have degree 2 and the nodes in $[n] \setminus I$ have degree 1. The bipartite graph is composed of $d \times d$- and $(d+1) \times (d+1)$- matchings.

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Proof.

- Statement holds for more general degree vectors.
- The polytope $\Delta_d \times \Delta_{n-1}$ is equidecomposable.
- The number of full-dimensional simplices equals the number of compositions of $n+d$.
- There is at most one cell per degree sequence.

□

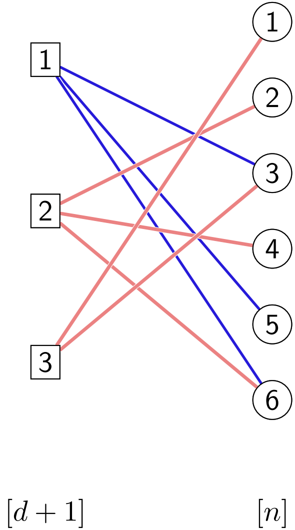
Signed Edges and Feasibility

Signed tropical matroid (L 2017)

Axiomatic description of combinatorics of a generalization of tropical linear inequality system (building on work by Ardila, Develin 2009, Oh, Yoo 2011, Horn 2012).

A distinguished subset of $[d+1] \times [n]$ (considered as edges) is negative (red). The other edges are positive (blue).

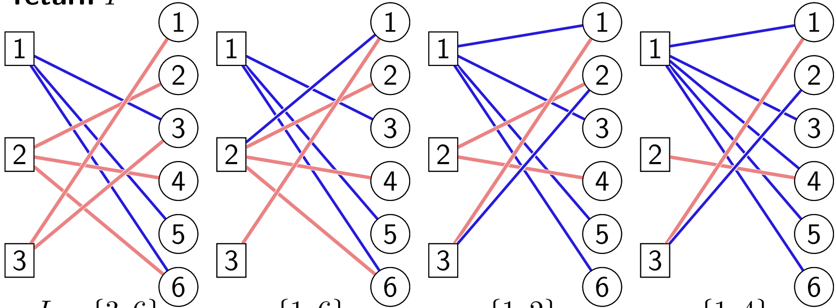
A Cramer covector is *feasible* if each node in $[n]$ is incident with a positive edge. Otherwise it is *infeasible*.



Abstract Tropical Feasibility Algorithm

Problem: Find feasible Cramer covector

$I \leftarrow$ appropriate d-elem. subset of $[n]$
while There is $j \in [n]$ only incident with negative edges in Cramer covector of I (and it is not totally infeasible) **do**
    $k \leftarrow$ node in $[n]$ incident with same node in $[d+1]$ via negative edge
    $I \leftarrow I \setminus \{k\} \cup \{j\}$
end while
return I



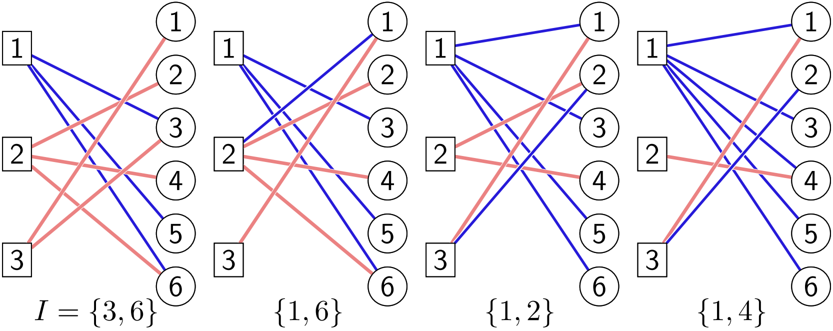
$I = \{3, 6\}$ $\{1, 6\}$ $\{1, 2\}$ $\{1, 4\}$

Abstract Tropical Feasibility Algorithm

Problem: Find feasible Cramer covector

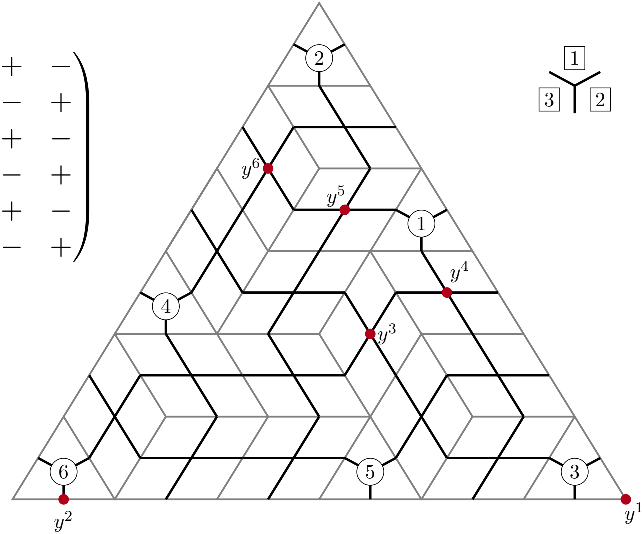
Theorem (L 2017)

The algorithm works in a not-necessarily regular triangulation of $\Delta_d \times \Delta_{n-1}$ and returns a feasible Cramer covector or a witness that there is no feasible Cramer covector.



TLP in a Non-Regular Triangulation of $\Delta_2 \times \Delta_5$

$$\begin{pmatrix} + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \end{pmatrix}$$



Ingredients

- Description of tropical inequality systems by covector graphs (JL 2016)
- Decomposition of tropical projective spaces with augmented (signed) covector graphs (JL 2016)
- Better understanding of the matching structure of covector graphs (JL 2016, L 2017)
- Combinatorial 'increase' lemma in not-necessarily regular triangulations of $\Delta_d \times \Delta_{n-1}$ (L 2017)
- Polyhedral constructions to deal with non-genericity and subpolytopes (De Loera, Rambau, Santos 2010, Horn 2012, ABGJ 2014, L 2017)

Runtime Analysis and the Secondary Fan of $\Delta_{d-1} \times \Delta_{n-1}$

Theorem (L 2017)

The algorithm takes $\mathcal{O}(d\omega)$ steps for a tropical linear inequality system given by a matrix $M \in (\mathbb{Z} \cup \{\infty\})^{n\times(d+1)}$.
Here, ω is the maximal finite entry of a fixed non-negative integer matrix which induces the same regular subdivision of $\Delta_d \times \Delta_{n-1}$ as M.

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Proof.

- Control flow of the algorithm depends only on the combinatorial structure of the subdivision.
- Coordinate increase in each step.

□

Conclusion and Further Work

Summary

- Extension of the theory of tropical covectors for inequality systems, boundary of tropical projective spaces, non-genericity.
- Tropical analogue of ‘oriented matroid programming’ as generalization of tropical linear programming.
- New tools to study the relation between classical and tropical linear programming.

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- Extension of the theory of tropical covectors for inequality systems, boundary of tropical projective spaces, non-genericity.
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Further Work

- Study 'long minimal integer vectors' in the cones of the secondary fan of $\Delta_{d-1} \times \Delta_{n-1} \quad \to$ hard instances.
- Investigate implications for classical linear programming.
- Matching structures.