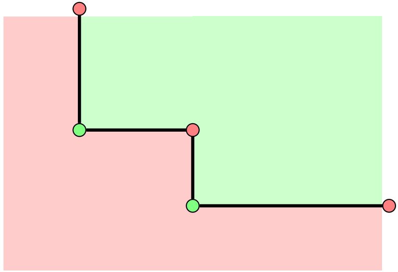
Monomial Tropical Cones for Multicriteria Optimization

Georg Loho



joint work with Michael Joswig

Multicriteria optimization

- multicriteria optimization problem:

$$\begin{array}{ll} \min & f(x) = \big(f_1(x), \ldots, f_d(x)\big) \\ \text{subject to} & x \in X \quad . \end{array}$$

- outcome space: $Z = f(X) \subseteq \mathbb{R}^d$
- $w \leqq z$: if $w_i \leq z_i$ for all $i \in [d]$, partial ordering on $\mathbb{R}^d$.
- for $S \subset \mathbb{R}^d$ the minimal elements with respect to $\leqq$ form the nondominated points.

Observation

The set dominated by $N \subseteq \mathbb{R}^d$ is exactly $\big(N + \mathbb{R}^d_{\geq 0}\big) \setminus N$.

We think of $Z = f(X)$ as a discrete set.

Tropical Geometry and Optimization

Interplay of Tropical Geometry and Optimization

- Geometric methods for mean-payoff games and complexity of classical linear programming through tropical linear programming (Akian et al. '12, Allamigeon et al. '14+)
- Computing tropical determinants by Hungarian method (Butkovič '10)

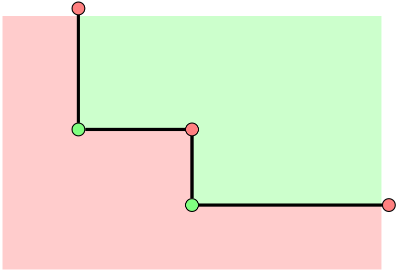
We want to enumerate all nondominated points.

Previous Work

- testing different regions of the search space (Sylva, Crema 2008)
- generating all n non-dominated points with $\mathcal{O}(n^d)$ scalarizations (Kirlik, Sayın 2014)
- assymptotically tight number of scalarizations $\Theta(n^{\lfloor d/2 \rfloor})$ (Dächert, Klamroth, Lacour, Vanderpooten 2016)

Key ideas

- Sets of the form $G + \mathbb{R}^d_{\geq 0}$ are tropically convex
- Tropical double description for 'monomial tropical cones' is an efficient algorithm to generate nondominated points
- Works for arbitrary number of objective functions
- Duality between 'local upper bounds' and nondominated points (local minima)



Tropical basics

Definition

Tropical numbers	$\mathbb{T}_{\max} = \mathbb{R} \cup \{-\infty\}$
Addition	$s \oplus t := \max(s, t) = -\min(-s, -t)$
Multiplication	$s \odot t := s + t$

Operations are extended componentwise to $\mathbb{T}_{\max}^{d+1}$

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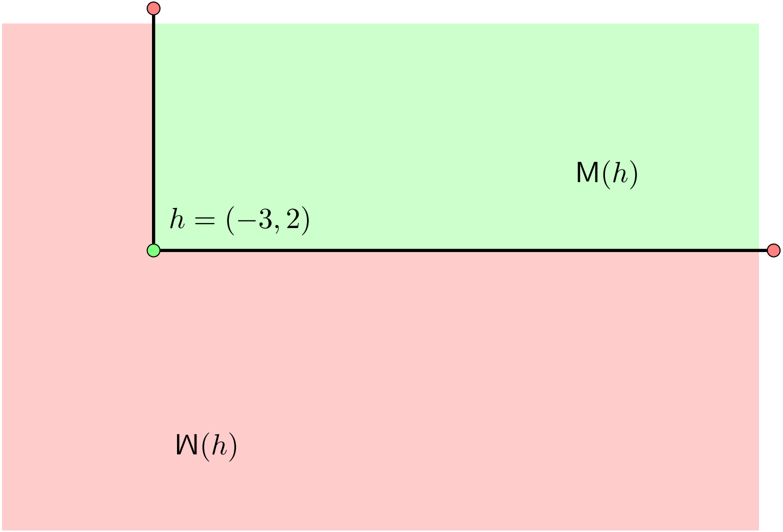
Example

$$(5 \oplus -7) \odot 10 \oplus -100 = 15$$

$(-3) \odot x \oplus 1 = 9$ valid for $x = 12$

But: $(-3) \odot x \oplus 9 = 9$ valid for every $x \leq 12$

Determining nondominated points with scalarizations



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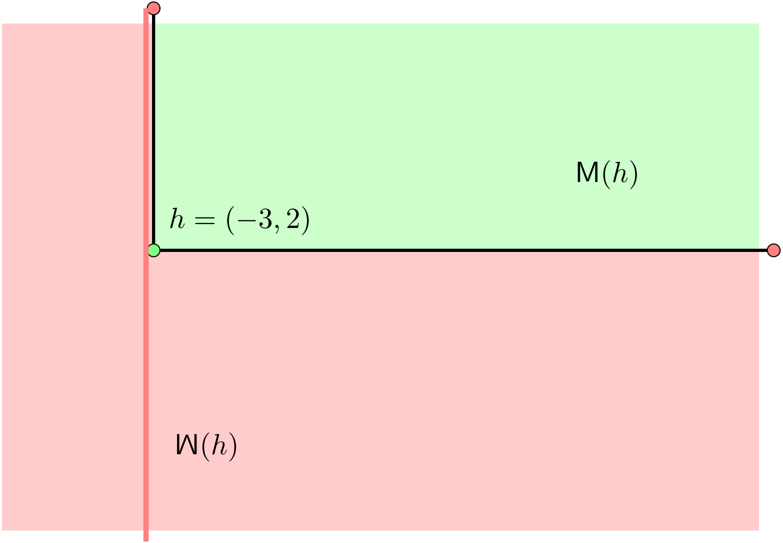

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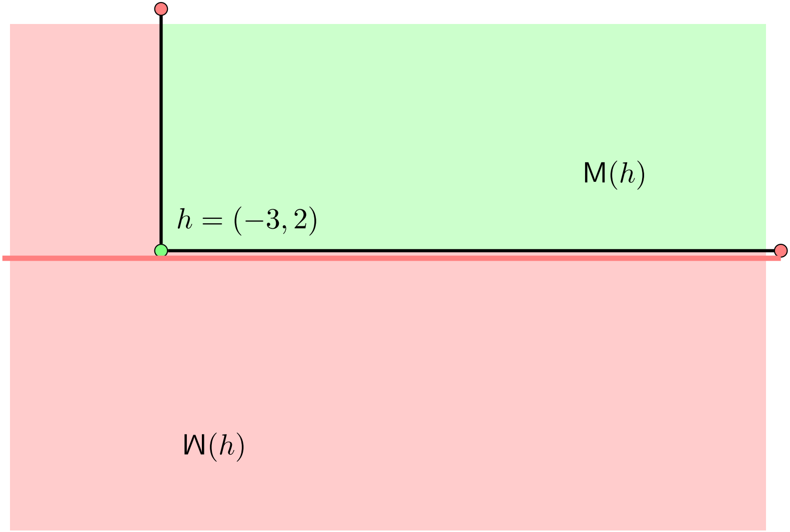

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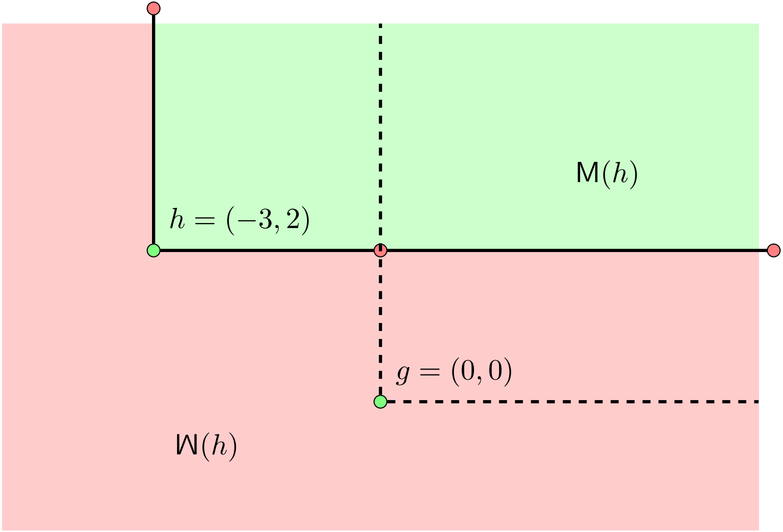

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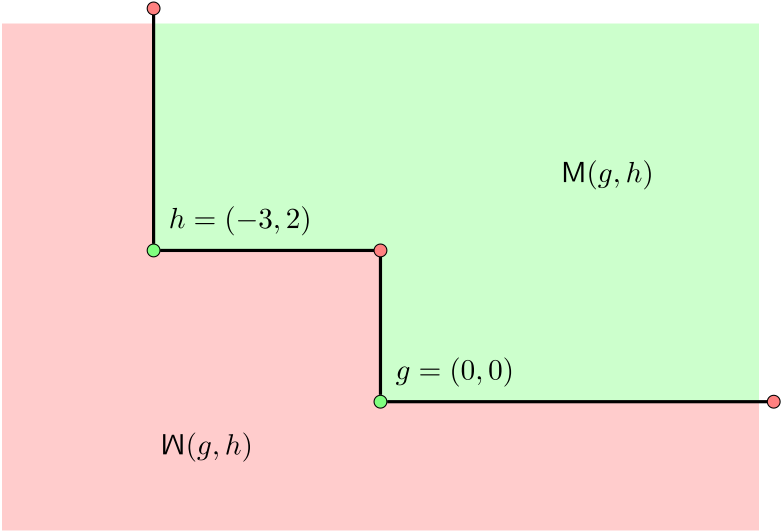

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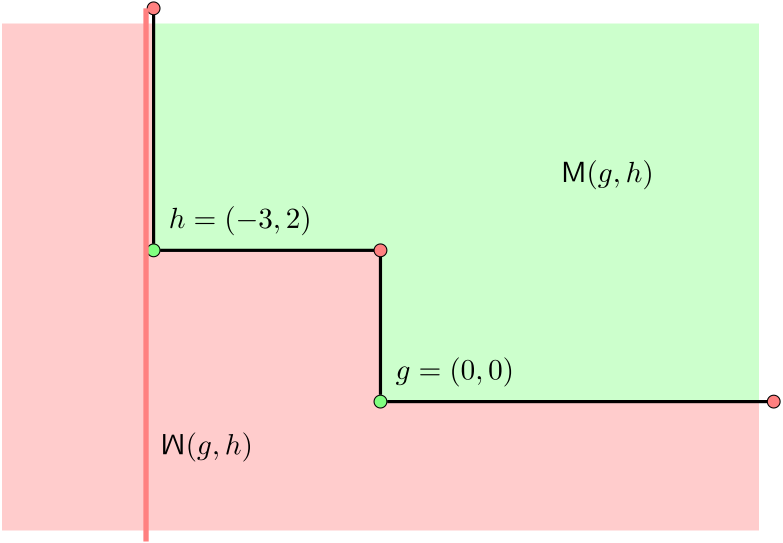

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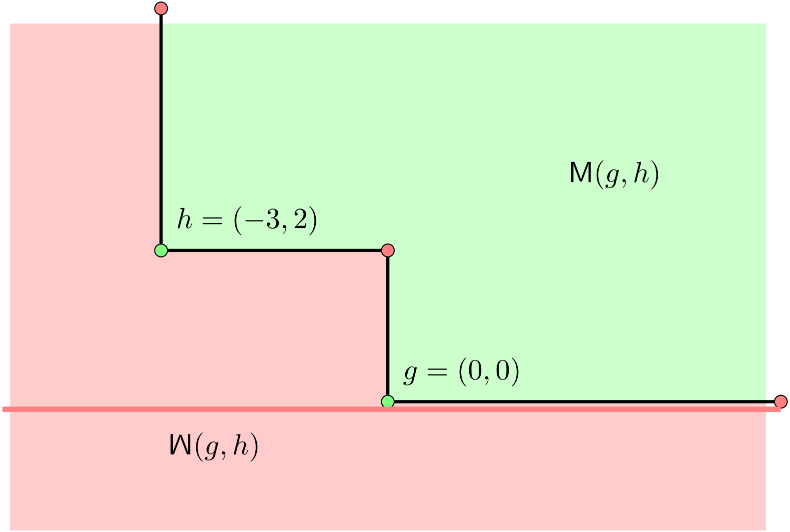

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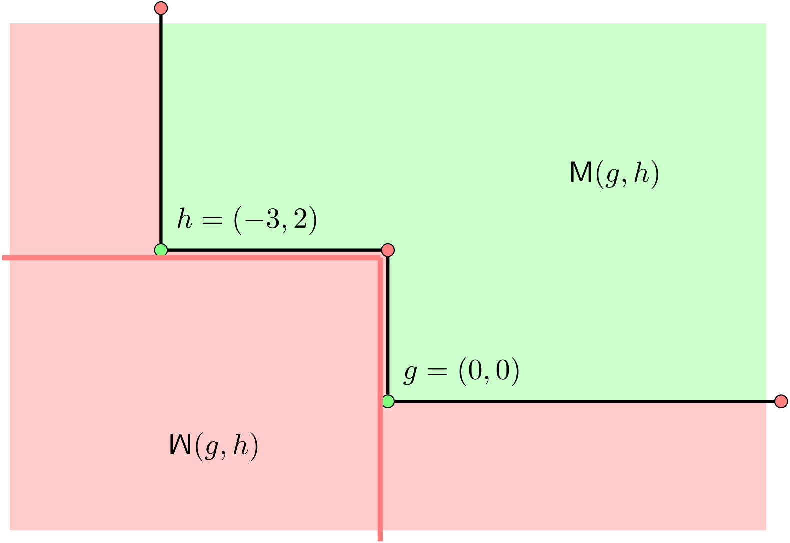


Adaptation of the generic method

(Klamroth et al. 2015, Joswig, L 2017)

Input: Image of the feasible set $Z \subset \mathbb{R}^d$

Output: The set of nondominated points.

$A \leftarrow \mathcal{E}_{\min} \cup e^{(0)}$
$G \leftarrow \emptyset$
$\Omega \leftarrow \mathcal{E}_{\min}$
while $A \neq \Omega$ **do**
    pick a in $A \setminus \Omega$
    $g \leftarrow \textsc{NextNonDominated}(Z, a)$
    if $g \neq$ None **then**
        $A \leftarrow \textsc{NewUpperBounds}(G, A, g)$
        $G \leftarrow G \cup \{g\}$
    else
        $\Omega \leftarrow \Omega \cup \{a\}$
    end if
end while
return G

Tropical Inner and Outer Description

- $\max$-tropical cone: subset $C \subseteq \mathbb{T}_{\max}^{d+1}$ with

$$\bigl(\max(\lambda + x_0, \mu + y_0), \ldots, \max(\lambda + x_d, \mu + y_d)\bigr) \in C$$

  for all $\lambda, \mu \in \mathbb{T}_{\max}$ and $x, y \in C$.
- G generates C if the latter is the minimal $\max$-tropical cone containing G; explicitly

$$\{\, \lambda_1 \odot g_1 \oplus \cdots \oplus \lambda_k \odot g_k \mid g_1, \ldots, g_k \in G, \lambda_1, \ldots, \lambda_k \in \mathbb{T}_{\max} \} \quad .$$

- extremal generators: minimal generating set consists of extremal generators; an element $g \in G$ is extremal iff, for $g_1, g_2 \in G$

$$g = \mu g_1 \oplus \nu g_2 \quad \Rightarrow \quad g = \mu g_1 \text{ or } g = \nu g_2 \quad .$$

Tropical Inner and Outer Description

- for vector $a \in \mathbb{T}_{\max}^{d+1}$ the set $\operatorname{supp}(a) = \{i \mid a_i \neq -\infty\}$ is its support.
- closed $\max$-tropical halfspace: a set of the form

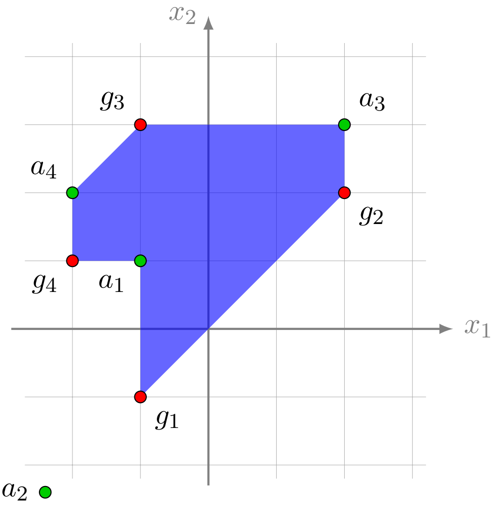
$$\left\{ x \in \mathbb{T}_{\max}^{d+1} \;\middle|\; \max(x_i + a_i \mid i \in I) \leq \max(x_j + a_j \mid j \in J) \right\}$$

for disjoint nonempty subsets $I, J \subset \operatorname{supp}(a)$

Theorem (Tropical Weyl-Minkowski Theorem (Gaubert, Katz 2007))

C *finitely generated* $\max$*-tropical cone* $\Leftrightarrow$
C *intersection of finitely many closed* $\max$*-tropical halfspaces.*

Tropical Inner and Outer Description



dehomogenized with $x_0 = 0$

$$G = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & -1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ \infty & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -2 & 2 \end{pmatrix}$$

$$x_0 \leq \max(x_1 + 1, x_2 - 1)$$
$$x_1 \leq x_2$$
$$\max(x_1 - 2, x_2 - 3) \leq x_0$$
$$\max(x_0, x_2 - 2) \leq x_1 + 2$$

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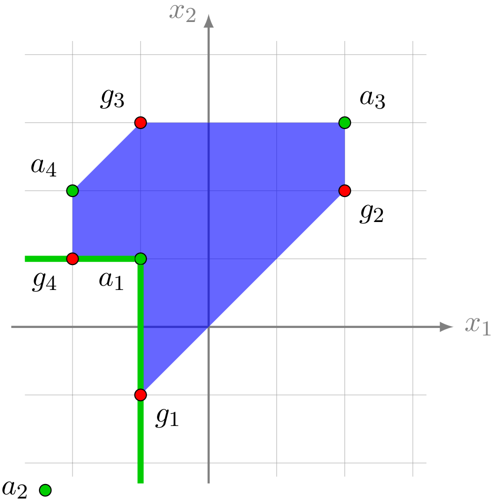

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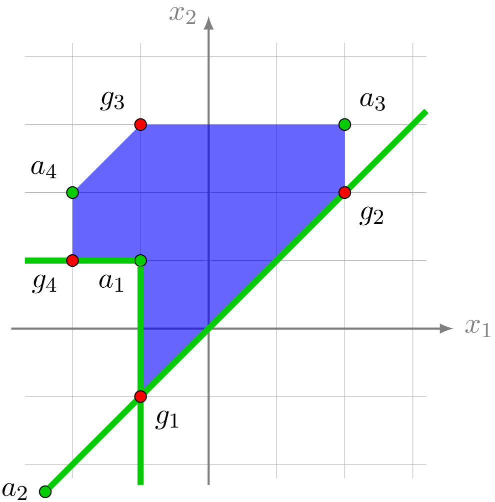

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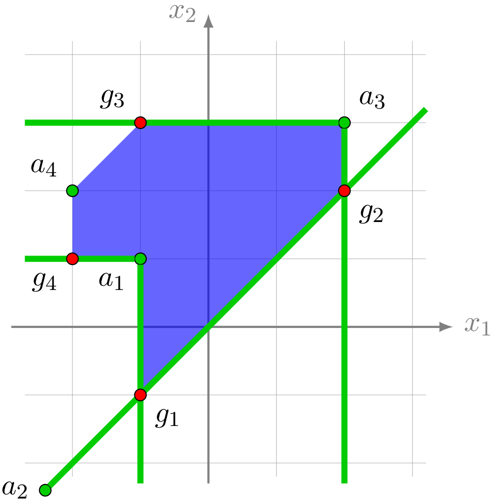

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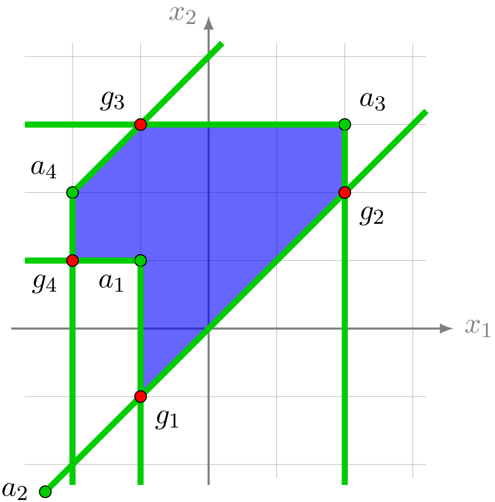

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Monomial tropical cones

Fix $G \subseteq \mathbb{T}_{\max}^{d+1}$.

- Monomial tropical cone $\mathsf{M}(G)$ (Allamigeon et al. 2010, JL 2017):

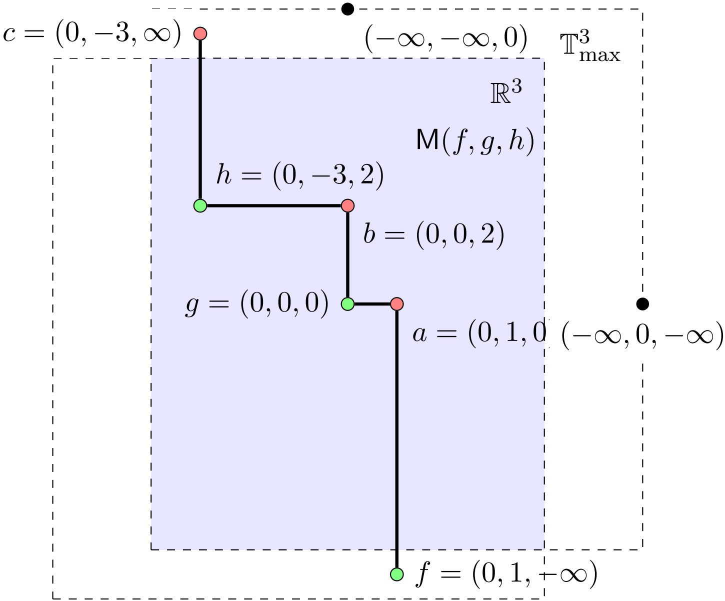
$$\bigcup_{g \in G} \left\{ x \in \mathbb{T}_{\max}^{d+1} \;\middle|\; x_0 - g_0 \leq \min(x_j - g_j \mid j \in \operatorname{supp}(g) \setminus \{0\}) \right\}$$

- $\mathsf{M}(G) = \overline{\mathsf{M}}(G) \cap \mathbb{R}^{d+1} =$

$$\left(\bigcup_{g \in G} g + \left(\{0\} \times \mathbb{R}^d_{\geq 0}\right) \right) + \mathbb{R} \cdot \mathbf{1}$$

- $\max$-tropical cone in $\mathbb{T}_{\max}^{d+1}$ generated by the finite set $G \cup \mathcal{E}_{\max}$, where $\mathcal{E}_{\max} = \{(-\infty, 0, -\infty, \ldots, -\infty), \ldots, (-\infty, -\infty, \ldots, -\infty, 0)\}$

Monomial tropical cones



Duality of monomial tropical cones

$\mathsf{W}(G)$: the complement of the interior of the $\max$-tropical cone $\mathsf{M}(G)$

$$\bigcap_{g \in G} \left(\mathbb{R}^{d+1} \setminus \left(g + \left(\{0\} \times \mathbb{R}^d_{>0} \right) \right) \right) + \mathbb{R} \cdot \mathbf{1}$$

$\overline{\mathsf{W}}(G)$: closure of $\mathsf{W}(G)$ in $\mathbb{T}^{d+1}_{\min}$ with respect to $\min$ (add points with ∞).

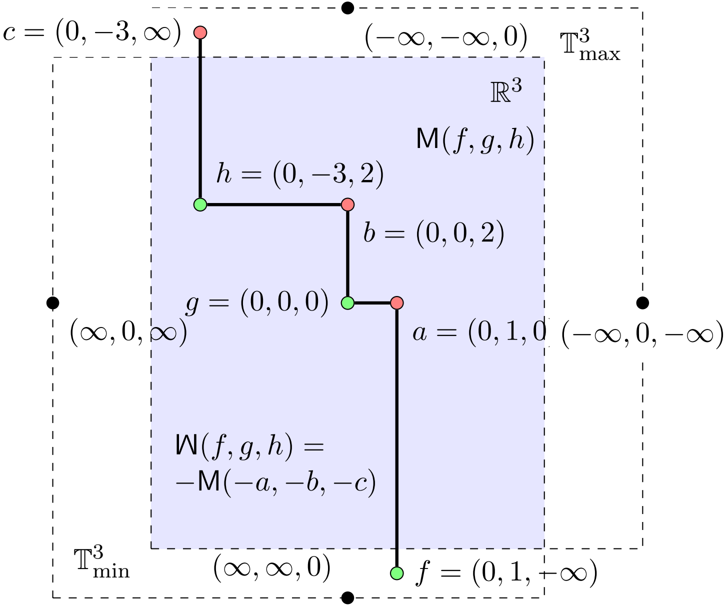
Theorem (Joswig, L 2017)

The set $\mathsf{W}(G)$ is a min-tropical cone in $\mathbb{R}^{d+1}$. More precisely, if $\mathcal{H}$ is a set of max-tropical halfspaces such that $\bigcap \mathcal{H} = \mathsf{M}(G)$, then

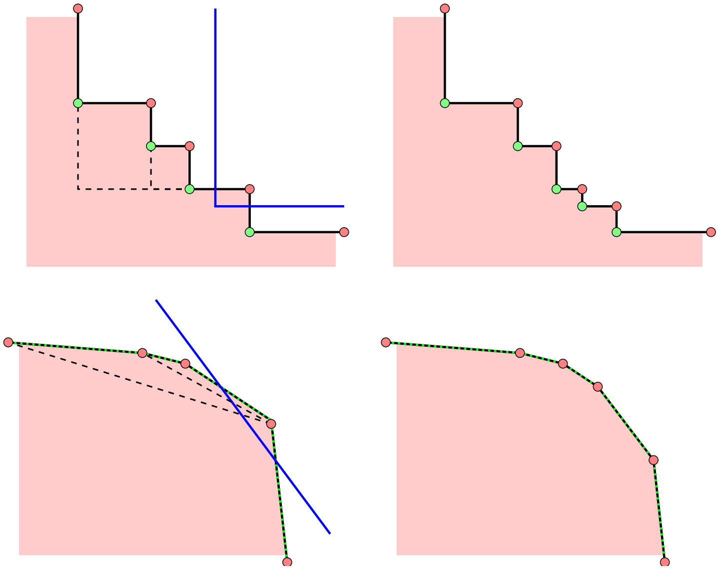
$$\mathsf{W}(G) \ = \ -\mathsf{M}(-A) \ ,$$

where $A \subset \mathbb{T}^{d+1}_{\min}$ is the set of apices of the tropical halfspaces in $\mathcal{H}$. In particular, the set $A \cup \mathcal{E}_{\min}$ generates $\mathsf{W}(G)$.

Duality of monomial tropical cones



Analogy of dual classical and tropical convex hull



Iterative tropical double description for monomial cones

Adaptation of 'tropical double description' by Allamigeon et al. (2010)
Input: A set $G \subset \mathbb{T}_{\max}^{d+1}$, the set A of extremal generators of $\mathsf{W}(G)$, and a point $h \in \mathbb{T}_{\max}^{d+1}$ with $h_0 = 0$.
Output: The set of extremal generators of $\mathsf{W}(G \cup \{h\})$.

- determine the set $A^{\geq}$ of points in A which fulfill the "monomial inequality" $x_0 \geq \min_{i \in [d]}(x_i - h_i)$ given by h
- initialize the set B of generators of $\mathsf{W}(G \cup \{h\})$ with $A^{\geq}$
- for each pair $(b, c) \in A^{\geq} \times \left(A \setminus A^{\geq}\right)$, determine the intersection of the tropical line through b and c with the boundary of the halfspace h
- discard the intersection points, which are not extremal in $\mathsf{W}(G \cup \{h\})$.

Upper bound theorems – many objective functions

Allamigeon et al. 2010, JL 2017

The number of extremal generators of a tropical cone in $\mathbb{T}_{\max}^{d+1}$ defined by n halfspaces is in $\mathcal{O}(n^{\lfloor d/2 \rfloor})$.

Proof

For each tropical cone there is a classical cone with similar combinatorial structure. Now claim follows from McMullen's Upper Bound Theorem.

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Theorem (JL 2017)

All n nondominated points of a discrete d-criteria optimization problem can be determined in $\Theta(n^{\lfloor d/2 \rfloor})$ with iterative tropical convex hull computation.

Conclusion

Summary

- generating all nondominated points with tropical convex hull
- study of monomial tropical cones and their duality
- promising connection between tropical algebra, multicriteria optimization and monomial ideals
- arxiv: 1707.09305