Tope Arrangements and Lattice Points of Simplices

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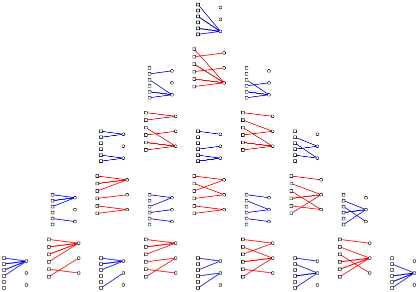


Oberseminar Darstellungstheorie, May 20th 2019
Joint with Ben Smith, QMUL

Topes

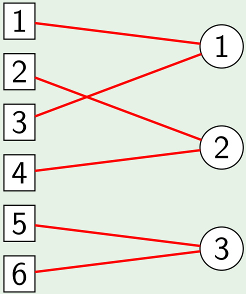
Let $v = (v_1, \ldots, v_d)$ be positive numbers with $\sum v_j = k \leq n$.

Definition

For a k-subset $\sigma \subseteq [n]$, a *tope* P_σ of type v is a bipartite graph on $[n] \sqcup [d]$ with

- left degree vector equal to e_σ, the indicator vector of σ
- right degree vector equal to its type.

Example (Tope of type $(2, 2, 2)$)



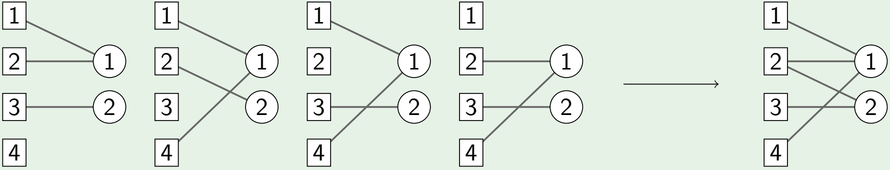
Tope Fields

Let $v = (v_1, \ldots, v_d)$ be positive numbers with $\sum v_j = k \leq n$.

Definition

A *tope field* $\mathcal{P} = (P_\sigma)$ of type v is a set of topes of the same type on the node sets $\sigma \sqcup [d]$, one for each k-subset $\sigma \subseteq [n]$.
A tope field is *linkage* if for every $(k+1)$-subset $\tau \subseteq [n]$, the union of the topes $L_\tau = \bigcup \{ P_\sigma \mid \sigma \subset \tau \}$ is a tree.

Example (Linkage tope field of type $(2, 1)$)



P_{123} P_{124} P_{134} P_{234} L_{1234}

Example: Matching Fields

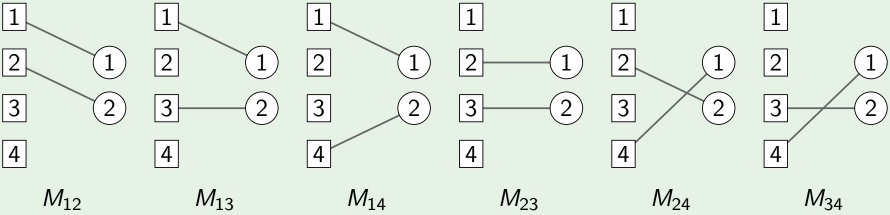
Let $K_{n,d}$ be the complete bipartite graph with node sets $[n]$ and $[d]$, where $n \geq d$.

Definition (Sturmfels, Zelevinksy '93)

A *matching field* $\mathcal{M} = (M_\sigma)$ is a set of perfect matchings on the node sets $\sigma \sqcup [d]$, one for each d-subset $\sigma \subseteq [n]$.

Example ($n = 4$, $d = 2$)

We have $\binom{n}{d} = 6$ matchings.

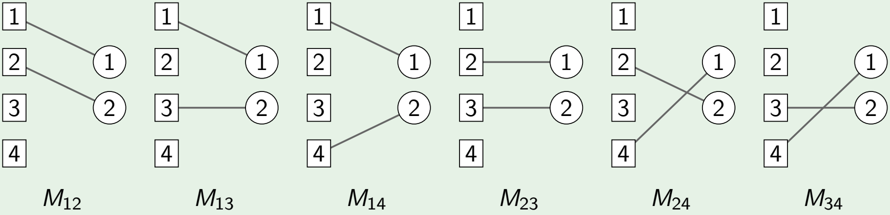


Coherent Matching Fields

Consider $K_{n,d}$ with weights given by an $n \times d$ matrix $W = (w_{ij})$.
For each $\sigma \subseteq [n]$, we can ask for the matching on $\sigma \sqcup [d]$ with maximal weight.
This gives the most natural class of matching fields called *coherent* matching fields.

Example

Consider the matrix of weights $W = \begin{bmatrix} 5 & 1 & 0 & 2 \\ 0 & 2 & 3 & 1 \end{bmatrix}$. The coherent matching field induced by W on $K_{4,2}$ is the same matching field as before.



Applications of (Coherent) Matching Fields

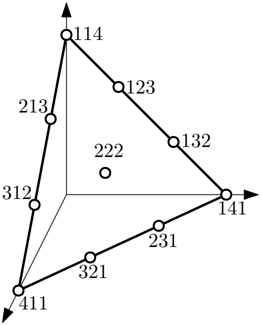
- Chow polytope of the variety $\{X \in \mathbb{C}^{m\times n} \mid \mathrm{rk}(X) < m\}$ $(m \leq n)$ (Sturmfels, Zelevinsky 1993)
- Tropical Grassmannian, Stiefel tropical linear spaces (Fink, Rincón 2015)
- Toric Degenerations of Grassmannians / Flag varietes / Schubert varieties from matching fields (Clarke, Mohammadi, Shaw 2018+)

Tope Arrangements

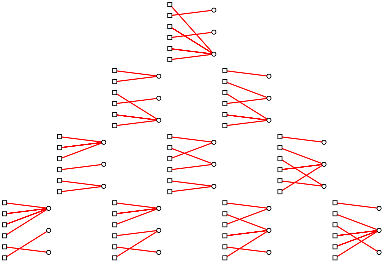
Definition

An *(n, d)-tope arrangement* is a collection of topes on $[n] \sqcup [d]$ such that:

- the right degree vectors are in bijection with the lattice points of $(n-d)\Delta_{d-1}$.
- if two topes contain a matching on a subset of nodes $J \sqcup I$, it is the same matching.



Lattice points of $3\Delta_2$

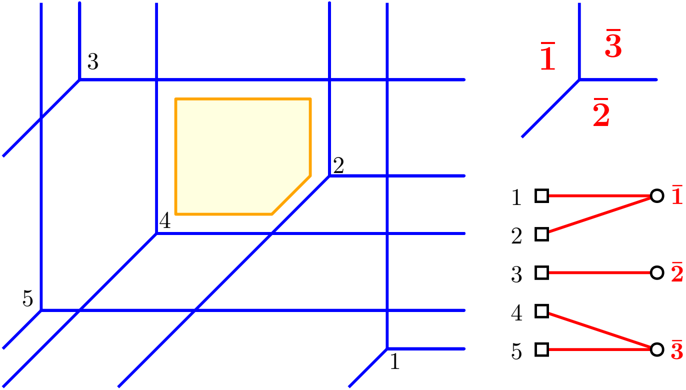


$(6, 3)$-tope arrangement

Example: Tropical Hyperplane Arrangements

A *tropical hyperplane* is a fan in $\mathbb{R}^{d-1}$ with d maximal cones, labelled by $\{1, \dots, d\}$.
An arrangement of n tropical hyperplanes decomposes $\mathbb{R}^{d-1}$ into regions.
Each region has a corresponding bipartite graph on $[n] \sqcup [d]$ with edges

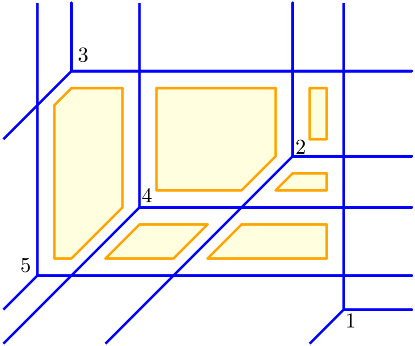
$$(j, i) \;\Leftrightarrow\; \text{the region is in the } i\text{-th cone of hyperplane } j$$

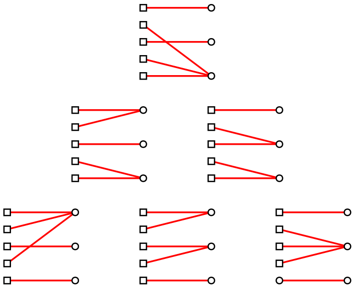


Example: Tropical Hyperplane Arrangements

Proposition (Ardila, Develin, Sturmfels '04,'09)

The bipartite graphs from the bounded regions of an arrangement of n tropical hyperplanes in $\mathbb{R}^{d-1}$ form an (n, d)-tope arrangement.





Tropical Hyperplane Arrangements	$\subset$	Subdivisions of product of simplices	$\subset$	Tope Arrangements

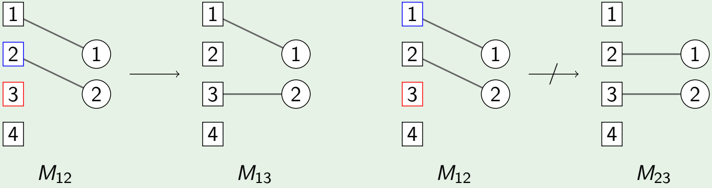
Linkage Matching Fields

Coherent matching fields are far more structured than arbitrary matching fields. In particular, they satisfy the *linkage property*:

Definition (Linkage Property)

For every $(d+1)$-subset $\tau \subseteq [n]$, the union of the matchings $L_\tau = \bigcup \{ M_\sigma \mid \sigma \subset \tau \}$ is a tree.

Example ($\tau = \{123\}$)

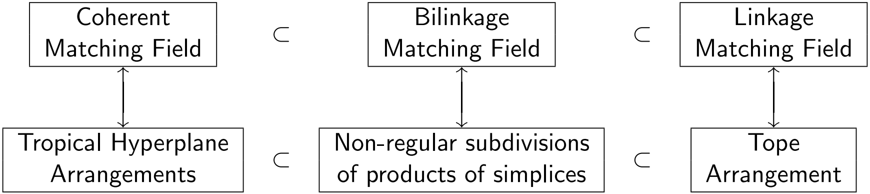


The trees L_τ arising this way are *linkage trees*.
A matching field that satisfies the linkage property is called a *linkage* matching field.

Tope Arrangements and Matching Fields

Theorem (L, Smith '18)

Tope arrangements and linkage matching fields are cryptomorphic.



Tope arrangements and linkage matching fields contain the same combinatorial information.

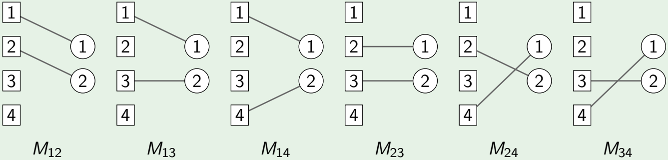
Chow Graphs

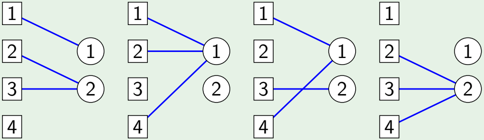
Sturmfels and Zelevinksy were interested in the Chow polytope $\mathrm{Ch}(\nabla_{\mathrm{d,n}})$ of $\nabla_{d,n}$. In studying this, they considered the following graphs:

Definition

Fix a matching field $\mathcal{M} = (M_J)$. A *Chow graph* Ω is a minimal bipartite graph such that $\Omega \cap M_J \neq \emptyset$ for all M_J.

Example

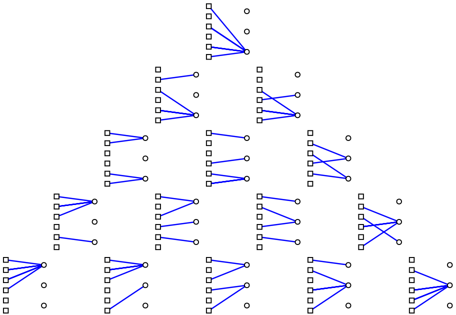




Chow Graphs and Lattice Points of Simplices

Conjecture (Sturmfels, Zelevinsky '93)

The Chow graphs of a linkage matching field are in bijection with the lattice points of $(n-d+1)\Delta_{d-1}$ via their right degree vector.

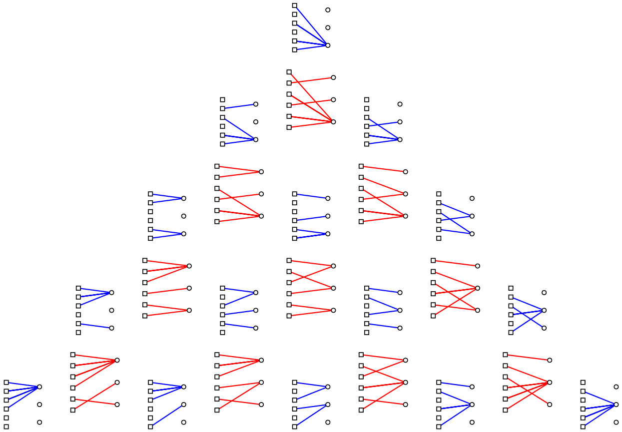


- Bernstein, Zelevinsky '93 - holds for coherent matching fields.
- L, Smith '18 - holds for all linkage matching fields.

Chow Graphs and Lattice Points of Simplices

Theorem (L, Smith '18)

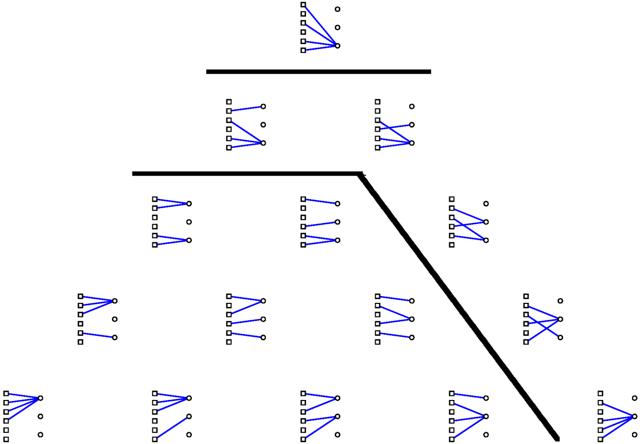
The Chow graphs of $\mathcal{M}$ can be recovered from the associated tope arrangement via intersections. This induces the bijection with $P_{n-d+1,d}$. Furthermore, they determine the tope arrangement via unions.



Chow Graphs and Lattice Points of Simplices

Theorem (Conjectured SZ '93)

A linkage matching field can be uniquely determined by the map from $(n-d+1)$-subsets to lattice points $(n-d+1)\Delta_{d-1}$ induced by the Chow graphs.

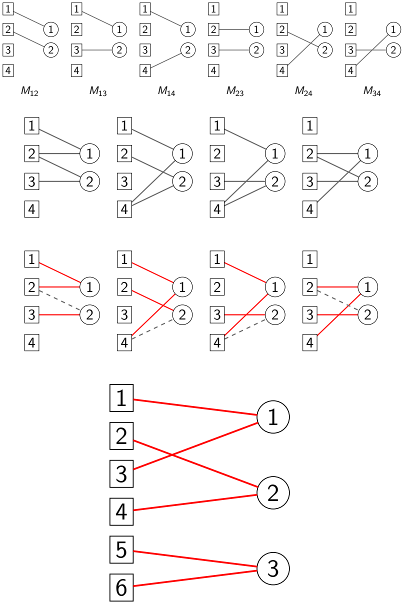


Amalgamation

A linkage tope field $\mathcal{P}$ of type v can induce a linkage tope field of type $v + e_j$.

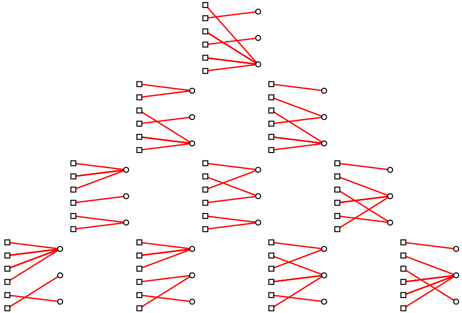
- Let $\mathcal{P}$ be of type v.
- Consider its set of linkage trees $\{ L_\tau \mid |\tau| = k + 1 \}$.
- Each L_τ contains a unique tope of type $v + e_j$.
- These form a linkage tope field.

This process is called *amalgamation*.



Iterated Amalgamation

- Starting from a linkage matching field $\mathcal{M}$, we can iterate amalgamation to build larger linkage tope fields.
- Eventually we get tope fields of type v such that $\sum v_j = n$.
- These tope fields consist of only one tope, that we call a *maximal tope*.
- The maximal topes are in bijection with lattice points of $(n-d)\Delta_{d-1}$ via their type (right degree vector).



Concluding Remarks

Summary

- Tope arrangements as new framework to study matching fields
- Lattice point bijections encoding the combinatorial data
- Connection with flag varieties

Reference: Matching fields and lattice points of simplices, Georg Loho and Ben Smith, arXiv:1804.01595, (2018)